Bachelor's degree thesis

Structural Characterization of Oscillatory Behavior in the Brain

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In partial fulfillment of the requirements for the

Bachelor's degree in Mathematics

Bachelor's degree in Aerospace Technology Engineering

July 2019

Abstract

Structural Characterization of Oscillatory Behavior in the Brain

by Robert Planas

Oscillations in the brain are one of the most pervasive and robust patterns found in the neural activity and they can be directly correlated with numerous cognitive phenomena. In this work, we study the structural network characteristics, in simple mean-field models of brain activity with bounded linear-threshold rate dynamics, that will ensure the lack of equilibria, and we numerically show that this approach is, indeed, a tight proxy for the existence of oscillatory behavior. Through a geometrical approach to the linear constructs intrinsic in the dynamics of the system, we provide multiple sets of sufficient conditions for the lack of equilibria in arbitrary multidimensional excitatory-inhibitory network (nE-mI). We further extend this characterization for the fully inhibitory. We also discuss the concept of degenerate oscillations, a tradeoff between competitive and cooperative behaviors involving different nodes or regions of the brain and we numerically compare prevalence of degenerate oscillations for all sets of sufficient conditions. Lastly, in the case of pairwise-unstable fully inhibitory networks, we further provide analytical characterizations of the degenerate oscillatory behavior in terms of network structure.

Keywords: Brain Networks, Oscillations, Excitatory-Inhibitory networks, Fully inhibitory networks, Degenerate Oscillations

Resumen

Caracterización estructural del comportamiento oscilatorio en en cerebro

por Robert Planas

Las oscilaciones en el cerebro son uno de los patrones más frecuentes y predominantes que se pueden encontrar en la actividad neuronal y están estrechamente relacionadas con múltiples fenómenos cognitivos. En la siguiente tesis, se estudian las características estructurales de las redes neuronales, usando modelos promedio de actividad cerebral con dinámicas lineales acotadas, que pueden garantizar la ausencia de equilibrio estable y se demuestra a través de simulaciones, que la caracterización de oscilaciones como tal es, efectivamente, una buena aproximación para definir la actividad oscilatoria. A través de un extenso estudio geométrico del conjunto de restricciones lineales intrínsecas en las dinámicas del sistema, se logra ofrecer múltiples conjuntos de condiciones suficientes que garantizan la ausencia de equilibrio estable en redes arbitrarias y multidimensionales de neuronas excitadoras e inhibitorias. Además, se extiende y se complementa esta caracterización para el caso particular de redes completamente inhibitorias.Se introduce y discute también el concepto de oscilaciones degeneradas, un punto de compromiso entre la competición y la cooperación entre distintos nodos o regiones del cerebro, y numéricamente, se compara como dicho comportamiento prevalece para redes arbitrarias y todos los conjuntos de condiciones suficientes propuestas. Finalmente, para el caso de redes completamente inhibitorias con comportamiento inestable dos a dos, se ofrecen caracterizaciones analíticas del comportamiento degenerado en términos de la estructura de la red.

Palabras clave: Redes cerebrales, Oscilaciones, Redes excitadoras-Inhibitorias, Redes completamente inhibitorias, Oscilaciones degeneradas

Acknowledgements

I would like to express my deepest gratitude to three different collectives of people.
First, to all my close relatives and friends, new and old ones. The process of research behind this thesis has had its bright moments and not so bright ones and people surrounding me has always been there to encourage me and to keep me on the right track.
Secondly, I would like thank CFIS, and with it, all the people involved, for giving me the great opportunity to pursue my interests in such a leading research group.
Lastly, but definitely not the least, I would like to say thank you to my three supervisors of this thesis. First, and as my contact with my home institution, UPC, I would like to thank Maria Alberich for her corrections, her help and for making the effort to read, comprehend and revise the whole extension of the thesis. Secondly, I would like to thank my supervisor in UCSD, professor Jorge Cortés, to accept me under his research group, give me the opportunity to attend lessons as one of his students and to become, for me, a reference on how balance between the academical and personal matters can be maintained while excelling in both of them. Finally, I would like to specially thank Erfan Nozari, who has had the patience during all these months to listen to my disorganized ideas, the kindness of correcting my reiterative mistakes and the enthusiasm of working beside me through the whole process.
To all of them, thank you!

Robert Planas Casadevall
Cassà de la Selva, July 2019

Preface

The research exposed on the following thesis has been conducted during a five months internship in the Department of Mechanical and Aerospace Engineering at the University of California San Diego under the tuition and supervision of both professor Jorge Cortés and one of his, recently graduated, phd students, Erfan Nozari. At the same time, Professor Maria Alberich from UPC has been the contact to my home institution and has also supervised the good development of this work.

Mainly, during my stay, I have been conducting research on linear threshold models and, focusing on its applications to brain networks with mean field rate dynamics, I have been trying to characterize the existence oscillatory behavior in the brain. Some of the results concluded on the behalf of excitatory and inhibitory networks will complete some of the already proposed ones from my supervisors, and all together, they will hopefully lead to a submission of a paper. However, this paper is still on its early stages and far more work, synthesis and coherence on all the result's line of discussion is still needed.

All my academical background at UPC has been specially useful to face all the challenges that this analysis on linear threshold models has supposed. Notions of algebra, matrix theory and graphs, algorithm implementation and numerical calculus, provided by my degree in Mathematics, and notions and understanding of computational analysis, control theory and basic and advanced physics, provided by my degree in Aerospace Technologies and Engineering have been indispensable in the satisfactory and correct development of the work.

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Chapter 1

Introduction and motivation for the problem

The brain has been object of study almost since the first philosophers tried to place the origin of thought and emotions. In the 18th century, when Luigi Galvani discovered that muscle contraction could be originated through an electrical stimulus, the first step to a more modern neuroscience was made. However, it was not until the 20th century, with Berger's groundbreaking discover of oscillatory activity in the brain [2], that the paradigm was able to change and a real progress towards the modern understanding of the brain was made. Since then, oscillations have been found to be present in a wide range of species and brain regions, and numerous studies have proven the correlations between these oscillations, namely its properties such as amplitude, phase, shape and coupling, and multiple brain and mental processes.

Despite the big importance oscillations display in cognitive phenomena, our understanding of them is limited and still far from being complete. Multiple approaches have been made in order to model and extract properties from oscillations. For example, a growing body of research has studied brain oscillations and its properties using models of phase oscillations, mainly with the Kuramoto model [3]. However, these phase-based models used to derive these properties remain deeply abstract and fundamental, and lack the capacity to explain or give origin to the oscillations. Linear threshold networks models, on the other hand, are indeed capable of modeling and explaining a wide range of non-linear phenomena.

With the goal of understanding this oscillatory behavior of the brain and giving insight to the existence of it, we will take an analytic approach and we will study the brain networks applying linear threshold dynamics with the aim to reveal the relationship between network structure and oscillatory activity.

In this line, our contributions are basically providing a detailed analysis of the dynamics of single networks with bounded linear threshold models. First, through an extensive geometrical approach to the intrinsic linear constrains of this bounded system, we derive sufficient conditions for the existence of lack of stable equilibria in terms of the network structure. We focus, mainly, on arbitrary excitatory-inhibitory networks and we show that the lack of stable equilibria can be used as a proxy for defining oscillatory behavior, being able to give explanation to almost an eighty

percent of the oscillation patterns found in the brain.
Secondly, we discuss the concept of degenerate oscillations, relating the so-called oscillation to a competition between the nodes that aim to oscillate. Finally, and motivated by the fact that gamma oscillations are correlated with mostly inhibitory networks [4], we study and derive sufficient conditions for the existence of oscillations in fully inhibitory networks. Together, these analytic and numeric results will provide great insights into the nature of brain oscillations and its relation to the structure of the underlying network.
In order to provide all the mentioned results, the following sections and structure will be developed. Firstly, the main problem will be exposed (2) by introducing the linear threshold dynamics model for brain networks and stating the objective of the whole research. Secondly, (3) oscillations in brain networks are going to be defined and numerically assumed or characterized to be equivalent as lack of stable equilibrium. Some assumptions to bound the problem are also going to be made. The next chapter, (4) will formalize the necessary and sufficient conditions for lack of stable equilibria, offering a set of inequalities that need to be both compatible and verified by the input of the system to provide instability. These inequalities will be further studied in a geometrical way in the subsequent section (5) and, with them, three sets of sufficient conditions for arbitrary networks will be provided (6). Then, a discussion based on numerical results on the degenerate rate for arbitrary networks and for each set of sufficient conditions will be exposed (7). Finally, results on the behalf of lack of stable equilibrium will be provided for pairwise unstable networks (8), fully inhibitory networks (9) and pairwise unstable fully inhibitory networks, where further conclusions on the degeneration of the system will be made.

Chapter 2

Problem Formulation

The following chapter will develop the analytic model of brain networks that will be used, the dynamics that will condition the problem and the equilibrium properties of this linear-threshold model. Furthermore, the whole notation that will be used in subsequent dicussions will be introduced

2.1 Dynamical Rate Models of Brain Networks

2.1.1 Average firing Rate

Neurons play the role of transmitters in brain networks, propagating signals among different regions. They do this by generating characteristic electrical pulses called action potentials or spikes. The transmission of information occur by firing specific sequences of spikes in different temporal patters, so we seek for a model that can account for the probability of a sequence of spikes being evoked.
An arbitrary spike sequence can be represented as a sum of infinitesimally narrow and idealized spikes, delta Dirac functions:

$$\rho(t) = \sum \delta(t - t_i) \tag{2.1}$$

With $t_i \in (0, T)$ and T being the trial time where the signal is recorded.
Let $\rho(t)$ be the neural response function, which can be used to re-express sums over spikes as integrals over time as follows:

$$\sum_{i=1}^{n} \delta(t - \tau_i) = \int_{-\infty} \infty \big(\delta(t - \tau)\rho(\tau)\big) d\tau \tag{2.2}$$

With this time dependent neural response function we can define the firing rate (r) as the count of action potentials that appear in a single trial. Then:

$$r = \frac{n}{T} = \frac{1}{T} \int_0^T \rho(\tau) d\tau \tag{2.3}$$

Averaging among different trials one would obtain the average firing rate $\langle r \rangle$.

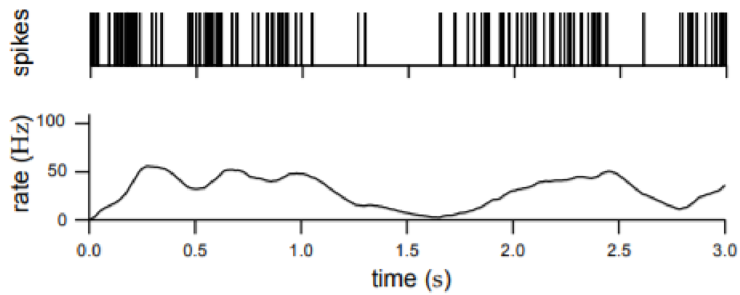


Figure 2.1: Representation of the neural response function or the spike train used for neuronal communication (top panel) and the corresponding (estimate of) firing rate (bottom panel) [1].

2.1.2 Linear-Threshold model with rate dynamics

Brain networks are composed of neurons, each receiving an electrical signal at its *dendrites*, from other neurons, and generating an electrical response to other neurons, at its *axon*. The transmission of activity from one neuron to another takes place at a *synapse*, so we will differ between pre-synaptic neurons and post-synaptic, and the synapse or transmission of information will occur from the pre-synaptic one to the post-synaptic. As exposed, the input and the output signals consists on a sequence of spikes or a neural response function, however in many brain areas the exact timing t_i of $\rho(t)$ is essentially random, and the information is encoded in the firing rate.
So, consider a pair of pre- and post-synaptic neurons with firing rates $x_{pre}(t)$ and $x_{post}(t)$. As a result of $x_{pre}(t)$, an electrical current $I_{post}(t)$ is produced in the post-synaptic neuron's dendrites. Assuming fast synaptic dynamics, $I_{post}(t) \propto x_{pre}(t)$, so let $w_{post,pre}$ be the characteristic proportionality constant, so $I_{post}(t) = w_{post,pre} x_{pre}(t)$.
The pre-synaptic neuron is called **excitatory** if $w_{post,pre} > 0$ and **inhibitory** if, otherwise, $w_{post,pre} < 0$. In other words, excitatory neurons will increase the activity of their out-neighbors while inhibitory neurons will decrease it. Notice that this is a property of neurons, not synapses, so a neuron either excites all its out-neighbors or inhibits them (Dale's law).
If the post-synaptic neuron receives input from multiple pre-synaptic neurons then, as linearity is present, $I_{post}(t)$ follows superposition law, so:

$$I_{post}(t) = \sum_j w_{post,j} x_j(t) \tag{2.4}$$

Where the sum computes all the pots-synaptic neighbors' influence.
The post-synaptic firing rate follows $x_{post}(t) = F(I_{post}(t))$,where F is a nonlinear "response function". In the literature, there are two widely used response functions, namely, sigmoidal and linear-threshold, and we will use the later, so if m_i is the maximum firing rate of the i-th neuron we have :

$$x_i = F[I_i(t)] = \max(0, \min(I_{post}(t), m_i)) \tag{2.5}$$

Which we will represent as $[\cdot]_0^{m_i}$ for single neurons or a population of neurons and $[\cdot]_0^{\boldsymbol{m}}$ for the complete network.

Finally, if $x_{post}(t)$ delays $[Ipost(t)]_0^{m_{post}}$ with a time constant τ_i we have:

$$\tau_{post}\dot{x}_{post}(t) = -x_{post}(t) + [I_{post}(t)]_0^{m_{post}} \tag{2.6}$$

This model has been extracted, and adapted to upper bound linear threshold, from [5].

2.1.3 Mean-field Linear-Threshold model with rate dynamics

Consider now a neural network composed by a large number of neurons, individually evolving according to (2.6). Since the number of neurons in a brain region is very large, consider a population of neurons with similar activation patterns as a single node and then consider the firing rate of this node to be the mean of the individual firing rates of the neuron population. Let $\boldsymbol{u} \in \mathbb{R}^N$ be the vector of average external (or background) inputs to the populations. By doing this and, under the standard assumptions (see [1], Ch 7), the mean-field dynamics of each the network can be described by a linear-threshold model like the following:

$$\tau\dot{\boldsymbol{x}}(t) = -\boldsymbol{x}(t) + [\boldsymbol{W}\boldsymbol{x}(t) + \boldsymbol{u}]_{\boldsymbol{0}}^{\boldsymbol{m}} \qquad \boldsymbol{x}(0) \in [\boldsymbol{0}, \boldsymbol{m}] \tag{2.7}$$

Where $\boldsymbol{x} \in \mathbb{R}^N$ is the state vector with x_i denoting the average firing rate of the i'th neural population, $\boldsymbol{W} \in \mathbb{R}^{N\times N}$ is the matrix of the average synaptic connectivity, $\boldsymbol{m} \in \mathbb{R}^N_{>0}$ is the vector of average maximum firing rates and $\tau > 0$ is the network time constant.

Problem 1. *For the bounded linear threshold network dynamics (2.7), characterize the relationship between network structure ($\boldsymbol{W}, \boldsymbol{m}$), the external input $\boldsymbol{u}$ and the non existence of stable equilibria in a single network.*

2.2 Oscillations in Linear-Threshold Networks

We will analyze the dynamics of (2.7) with the aim of finding conditions on the network structure ($\boldsymbol{W}, \boldsymbol{m}$) that can lead to oscillatory behavior. Dale's law [1], which states that a neuron cannot excite some of its postsynaptic targets and inhibit others, conditions the structure of $\boldsymbol{W}$ as a matrix where each column can only be nonnegative or non-positive. We aim for the study for arbitrary networks with n excitatory nodes and m inhibitory nodes. Let $N = n+m$, $n \geq 1$, $m \geq 1$, and then consider:

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ \boldsymbol{c} & -\boldsymbol{d} \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_e} \\ \boldsymbol{u_i} \end{bmatrix}, \quad \boldsymbol{m} = \begin{bmatrix} \boldsymbol{m_e} \\ \boldsymbol{m_i} \end{bmatrix}, \tag{2.8}$$

where $\boldsymbol{a} \in \mathbb{R}^{n\times n}_{\geq 0}$, $\boldsymbol{b} \in \mathbb{R}^{n\times m}_{\geq 0}$, $\boldsymbol{c} \in \mathbb{R}^{m\times n}_{\geq 0}$, $\boldsymbol{d} \in \mathbb{R}^{m\times m}_{\geq 0}$. In order to study the equilibria of (2.7), we represent it as a switched affine system so we decompose $\mathbb{R}^N$ into 3^N switching regions $\{\Omega_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma}\in\{0,l,s\}^N}$, where 0, l and s denote inactive, active (linear) and saturated nodes, respectively. These regions are defined by:

$$\boldsymbol{x} \in \Omega_{\boldsymbol{\sigma}} \Leftrightarrow \begin{cases} (\boldsymbol{W}\boldsymbol{x}+\boldsymbol{u})_i \in (-\infty, 0]; & \forall i \ s.t. \ \sigma_i = 0, \\ (\boldsymbol{W}\boldsymbol{x}+\boldsymbol{u})_i \in [0, m_i]; & \forall i \ s.t. \ \sigma_i = l, \\ (\boldsymbol{W}\boldsymbol{x}+\boldsymbol{u})_i \in [m_i, +\infty; & \forall i \ s.t. \ \sigma_i = s, \end{cases}$$

Taking this into account, (2.7) can be rewritten as:

$$\tau\dot{\boldsymbol{x}}(t) = (-\boldsymbol{I} + \boldsymbol{\Sigma}^l\boldsymbol{W})\boldsymbol{x} + \boldsymbol{\Sigma}^l\boldsymbol{u} + \boldsymbol{\Sigma}^s\boldsymbol{m} \qquad \forall \boldsymbol{x} \in \Omega_{\boldsymbol{\sigma}}, \boldsymbol{\sigma} \in \{0, l, s\}^N \tag{2.9}$$

where for any $\boldsymbol{\sigma}$, $\boldsymbol{\Sigma}^l \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $\Sigma^l_{i,i} = 1$ if $\sigma_i = l$ and $\boldsymbol{\Sigma}^s \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $\Sigma^s_{i,i} = 1$ if $\sigma_i = s$, and $\boldsymbol{m}$ is the saturation vector, both positive and negative. Each $\boldsymbol{\sigma}$ has a corresponding equilibrium candidate $\boldsymbol{x}^*_{\boldsymbol{\sigma}} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l\boldsymbol{W})^{-1}(\boldsymbol{\Sigma}^l\boldsymbol{u} + \boldsymbol{\Sigma}^s\boldsymbol{m})$ and the equilibria of (2.7) are the $\boldsymbol{x}^*_{\boldsymbol{\sigma}}$ that belong to their respective switching regions $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$.

2.3 Notation

In order to understand and follow most of the discussions and results presented on the behalf of lack of stable equilibrium, some notations need to be introduced.

- $\boldsymbol{\sigma} = \{0, s, l\}^N$ will be used to refer to a concrete state of the nodes and $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ will be used to represent the actual region on the state space where the nodes are on the states $\boldsymbol{\sigma}$.
- $\boldsymbol{\sigma}'$ will be used to represent the state of the excitatory nodes while $\boldsymbol{\sigma}''$ will refer to the inhibitory ones. Then $\boldsymbol{\sigma} = (\boldsymbol{\sigma}', \boldsymbol{\sigma}'')$.
- E will represent the index set of excitatory nodes.
- I will represent the index set of inhibitory nodes.
- $\Pi_{\boldsymbol{\sigma}}$ is going to be the corresponding permutation matrix to $\boldsymbol{\sigma}$, that will match every node in linear state to the right bottom block of $\boldsymbol{W}$. In matrix form it can be seen as:

$$\boldsymbol{\Pi}_{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{I}_n & 0 \\ 0 & \boldsymbol{\Pi}_{\boldsymbol{\sigma}} \end{bmatrix}$$

That applied to the system (2.8) give $\boldsymbol{W}_\Pi$ and $\boldsymbol{u}_\Pi$ which are the permuted versions of both $\boldsymbol{W}$ and $\boldsymbol{u}$:

$$\boldsymbol{W}_\Pi = \boldsymbol{\Pi}_{\boldsymbol{\sigma}}\boldsymbol{W}\boldsymbol{\Pi}^T_{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{a} & -\boldsymbol{b}\boldsymbol{\Pi}^T_{\boldsymbol{\sigma}''} \\ \boldsymbol{\Pi}_{\boldsymbol{\sigma}''}\boldsymbol{c} & -\boldsymbol{\Pi}_{\boldsymbol{\sigma}''}\boldsymbol{d}\boldsymbol{\Pi}^T_{\boldsymbol{\sigma}''} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & -\boldsymbol{b}_\Pi \\ \boldsymbol{c}_\Pi & -\boldsymbol{d}_\Pi \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & -\boldsymbol{b}_\Pi^{\{0,s\}} & -\boldsymbol{b}_\Pi^{\{l\}} \\ \boldsymbol{c}_\Pi^{\{0,s\}} & -\boldsymbol{d}_\Pi^{\{0,s\},\{0,s\}} & -\boldsymbol{d}_\Pi^{\{0,s\},\{l\}} \\ \boldsymbol{c}_\Pi^{\{l\}} & -\boldsymbol{d}_\Pi^{\{l\},\{0,s\}} & -\boldsymbol{d}_\Pi^{\{l\},\{l\}} \end{bmatrix}$$

$$\boldsymbol{u}_\Pi = \boldsymbol{\Pi}_{\boldsymbol{\sigma}}\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_\Pi^e \\ \boldsymbol{u}_\Pi^{\{0,s\},i} \\ \boldsymbol{u}_\Pi^{\{l\},i} \end{bmatrix}$$

- Σ^s will be a diagonal matrix representing the nodes in positive saturation state such that $\Sigma_{i,i} = 1$
  $Leftrightarrow \sigma_i = s$. $\Sigma^{s,i}$ and $\Sigma^{s,e}$ will be sub-matrices of Σ^s representing only the inhibitory nodes and the excitatory nodes respectively.

- Σ^l and Σ^0 will be the corresponding with linear state and negative saturation state. We will also define $\Sigma^{l,e}$, $\Sigma^{l,i}$ and $\Sigma^{0,e}$, $\Sigma^{0,i}$ to refer to the excitatory or inhibitory nodes only.
- Let $\mid \Sigma^{*,i} \mid$ represent the number of nodes found in the concrete state.
- It follows that $|\Sigma^{0,i}| + |\Sigma^{l,i}| + |\Sigma^{s,i}| = m$ and that $|\Sigma^{0,e}| + |\Sigma^{l,e}| + |\Sigma^{s,e}| = n$.
- Without loss of generality, let's consider that the excitatory nodes are always the top left coefficient of the network structure.
- Let $(\boldsymbol{I} - \Sigma^l \boldsymbol{W})^{-1}$ be described as $K^l_{i,j}$, and let $|W|_{i,j} = |\boldsymbol{W}|_{i,j}$ for the shake of simplicity. Following the described notation, the matrices involved in the problem are:

$$|W| = \begin{bmatrix} \mid W_{1,1} \mid & \mid W_{1,2} \mid & \dots & \mid W_{1,m+1} \mid \\ \mid W_{2,1} \mid & \mid W_{2,2} \mid & \dots & \mid W_{2,m+1} \mid \\ \vdots & \ddots & \dots & \vdots \\ \mid W_{m+1,1} \mid & \mid W_{m+1,2} \mid & \dots & \mid W_{m+1,m+1} \mid \end{bmatrix} \quad (\boldsymbol{I} - \Sigma^l \boldsymbol{W})^{-1} = \begin{bmatrix} K_{1,1} & \dots & K_{1,m+1} \\ \vdots & \ddots & \vdots \\ K_{m+1,1} & \dots & K_{m+1,m+1} \end{bmatrix} \tag{2.10}$$

  Note that the matrix $(\boldsymbol{I} - \Sigma^l \boldsymbol{W})^{-1}$ will be different from identity only on those i-th rows where $(\Sigma^l)_{i,i} = 1$.

- Finally, let $\boldsymbol{m}_{\Pi,\boldsymbol{\Sigma}^s}$ be:

$$\boldsymbol{m}_{\Pi,\boldsymbol{\Sigma}^s} = \boldsymbol{\Pi_\sigma} \boldsymbol{\Sigma}^s \boldsymbol{m} = \begin{bmatrix} m^e_{\Pi,\boldsymbol{\Sigma}^s} \\ m^i_{\Pi,\boldsymbol{\Sigma}^s} \\ 0 \end{bmatrix}$$

Chapter 3

Axiom and assumptions

The following chapter will discuss the main axioms and assumptions that will drive the results and conclusions developed through the whole thesis. The concept of oscillations in linear threshold models will be introduced and the two different ways oscillations can occur will be discussed and quantified.
Finally, for the chosen approach of study, networks with lack of stable equilibria, some assumptions to simplify and branch the problem will be made.

3.1 Definition of oscillations

A notion to characterize and define oscillatory behavior in a linear threshold model is needed. It has been observed that any non-stable temporal evolution of the so-called system presents some regular patterns which repeat periodically. For this reason, we are going to measure the existence of oscillations using this notion of regularity and characterize these repeating patterns as oscillations. We construct a regularity index as follows.
Let $X(f)$ be the Fourier Transform of $x(t)$ and $f_{max} = argmax_f|X(f)|$. Then:

$$\chi_{reg} = \frac{|X(f_{max})|}{\max\{\big(X((1-\varepsilon)f_{max})\big), \big(X((1+\varepsilon)f_{max})\big)} \in [1, \infty) \tag{3.1}$$

where $\varepsilon \in (0, 1)$.
Note that χ_{reg} captures the peak on the power spectra and its value will indicate the regularity of the oscillation. A value of $\chi_{reg} = 1$ indicates a flat spectrum (lack of oscillations) while as $\chi_{reg} \to \infty$ it becomes a periodic oscillation.
In practice a value of $\chi_{reg} \geq 2$ for $\varepsilon \leq 0,1$ is enough to capture the oscillatory behavior with more regularity as χ_{reg} grows. This specific definition of χ_{reg} and its application on the characterizing the oscillatory behavior in the brain has been extracted form [6].
We will use χ_{reg} to determine the existence of oscillations.

3.2 Oscillations and lack of equilibria

Oscillations in a linear threshold model $(\boldsymbol{W}, \boldsymbol{m}, \boldsymbol{u})$ can steer from either the attraction of a limit cycle (not necessarily perfectly periodic) or the lack of stable equilibria in the network, which creates chaotic periodical patterns. The main results on this thesis are on the behalf of the lack of stable equilibria. So, the question of in which extend the characterization of networks with lack of equilibria would capture the totality of the oscillatory behavior in this kind of models needs to be answered.
In order to do it, multiple networks have been randomly generated and later analyzed using the following criteria:

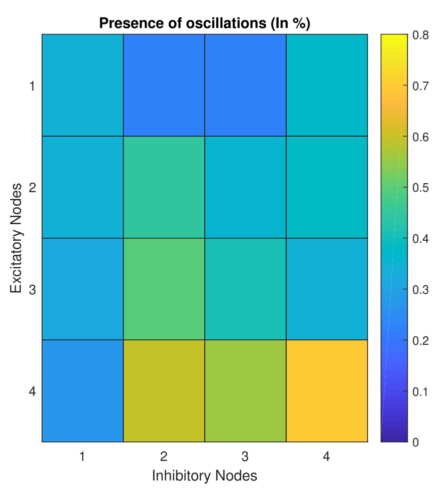
$$\boldsymbol{W} = \begin{cases} W_{i,k} = \mathcal{U}(0, a_{max}) & If k \in \{1, \dots, n\} \text{ and } \forall i \\ W_{i,k} = \mathcal{U}(-d_{max}, 0) & If k \in \{1, \dots, n\} \text{ and } \forall i \end{cases} \tag{3.2}$$

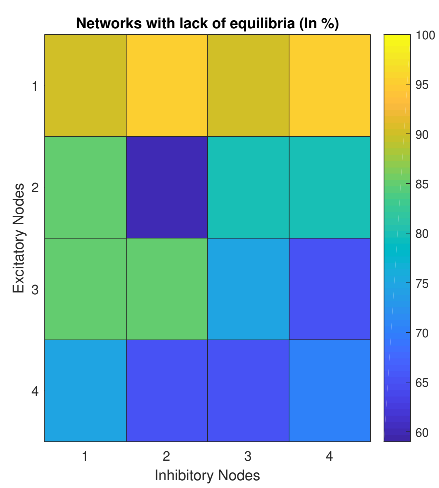
$$\boldsymbol{m} = \Big\{ m_i = \mathcal{U}(0, m_{max}) \quad \forall i \qquad \boldsymbol{u} = \Big\{ u_i = \mathcal{U}(-u_{min}, u_{max}) \quad \forall i \tag{3.3}$$

Where $a_{max}, d_{max}, m_{max} \in \mathbb{R}_{\geq 0}$ and u_{max} and u_{min} are choose to be:

$$\begin{cases} u_{max} = mean(|\boldsymbol{W}|)mean(\boldsymbol{m})m \\ u_{min} = mean(|\boldsymbol{W}|)mean(\boldsymbol{m})n \end{cases} \tag{3.4}$$

For each network, multiple initial conditions have been considered with the aim of finding oscillations. For those networks where oscillations were found, whether they steer from lack of equilibria of from the attraction of a limit cycle has been check. Furthermore, for those where the oscillatory behavior was due to a limit cycle, how big was the region of attraction covered by limit cycles, for that system, has also been quantified.
The results are the following:





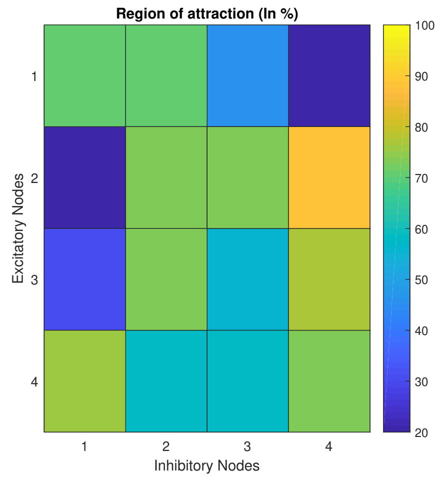


Figure 3.1: From top to bottom to right. Presence of oscillations in arbitrary networks, the average value is 0.3970 %, increasing the presence as the network becomes larger. Percentage of networks displaying oscillations due to lack of stable equilibria, the average value is 78 % decreasing as the network becomes larger. Region of attraction of limit cycles. The average value is 59.40%.

Results 3.2.1. *On average, only about the 0.4% of networks present oscillatory behavior. For those oscillating, about a 78% of the oscillations steer from lack of stable equilibria and the remaining 22% from a limit cycle. Furthermore, for those limit cycle oscillations, about the 60% of initial conditions will converge to limit cycles, not necessarily always the same, while the remaining 40% will converge to some stable equilibrium point.*

As it has been numerically shown, oscillatory behavior steers mostly from the lack of stable equilibiria, at least for networks up to eight nodes. As the networks becomes larger, limit cycles become more recurrent.
We will characterize oscillations exclusively through the condition of lack of stable equilibira, assuming that it is only a proxy (partial characterization) to this behavior as it does not capture the totallity of it. However, we can say that it is, indeed a tight proxy because we can confidently say that it can explain and give origin to almost the eighty percent of the oscillation behavior taking place in linear threshold models.

Assumption 1. *The oscillatory behavior in brain networks with mean-field linear-threshold with rate dynamics will be equivalently treated as lack of stable equilibrium, despite of knowing that it is only a proxy, partial explanation, to it.*

Remark 3.2.2. *The validity of the proxy might be even more than the one concluded after the simulations. Due to numerical errors, some stable equilibrium points are mistaken as limit cycle (the 1E-1I networks should present oscillations only if there is lack of stable equilibria and that is not the case. ([6])*

3.3 Instability of excitatory nodes in linear state

In order to bound the problem and simplify the results we are going to make assumptions on the instability of the excitatory nodes.

Assumption 2. ***(Unstability of the switching regions were one excitatory node is in linear mode).***
For each switching region $\boldsymbol{\Omega_\sigma}$ where $\exists \sigma_i = l$ and $i \in \{1, \ldots, n\}$, meaning that i is an excitatory node, we will ***assume*** *that the dynamics of the system:*

$$-\boldsymbol{I} + \boldsymbol{\Sigma}^l \boldsymbol{W} \tag{3.5}$$

have at least one eigenvalue λ_i for which $Re(\lambda_i) > 0$, i.e. the equilibrium point of this switching region is unstable.

Chapter 4

Networks with lack of stability

As it has been discussed on chapter (3), the characterization of oscillations will be developed on the regard of the lack of stability in the network, as this have been numerically proved to be a good proxy to define the oscillatory behavior in the brain.
On this behalf we will define the input regions that provide this lack of stable equilibrium and the set of inequalities that need to be both compatible and verified.

4.1 Description of regions without equilibria

Theorem 4.1.1. ***(Networks with lack of stability).***
Consider the dynamics (2.7)-(2.8) and assume $\boldsymbol{W}$ verifies assumption (2). Then, the network does not have any stable equilibria if

$$\boldsymbol{u} \in \mathbb{R}^N \setminus Y, \tag{4.1}$$

where

$$Y = \bigcup_{\boldsymbol{\sigma}' \in \{0,s\}^n} \bigcup_{\boldsymbol{\sigma}'' \in \{0,s,l\}^m} \Big(\bigcap_{i=1}^{N} y_{\boldsymbol{\sigma}',\boldsymbol{\sigma}'',i} \Big)$$

and, assuming that $\exists (\boldsymbol{I} + \boldsymbol{d}_{\Pi}^{\{l\},\{l\}})^{-1}$ then

$$y_{\boldsymbol{\sigma}',\boldsymbol{\sigma}'',i} = \begin{cases} \boldsymbol{u} \mid (\boldsymbol{H}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \geq m_i & \forall i \in \{1,\dots,n\} \ s.t. \ \sigma_i = s \\ \boldsymbol{u} \mid (\boldsymbol{H}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \leq 0 & \forall i \in \{1,\dots,n\} \ s.t. \ \sigma_i = 0 \\ \boldsymbol{u} \mid (\boldsymbol{F}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \geq m_i & \forall i \in \{n+1,\dots,n+m-t\} \\ & \quad s.t. \ \sigma_i = s \\ \boldsymbol{u} \mid (\boldsymbol{F}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \leq 0 & \forall i \in \{n+1,\dots,n+m-t\} \\ & \quad s.t. \ \sigma_i = 0 \\ \boldsymbol{u} \mid ((\boldsymbol{G}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \leq m_i) & \forall i \in \{n+m-t+1,\dots,n+m\} \\ \quad \cap ((\boldsymbol{G}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \geq 0) & \quad s.t. \ \sigma_i = l \end{cases} \tag{4.2}$$

Where, for a given $\boldsymbol{\sigma}$, $\boldsymbol{H}_{\boldsymbol{\sigma}}$, $\boldsymbol{F}_{\boldsymbol{\sigma}}$, $\boldsymbol{G}_{\boldsymbol{\sigma}}$ are:

$$
\begin{aligned}
\boldsymbol{H}_{\boldsymbol{\sigma}}(\boldsymbol{u}) &= \boldsymbol{a}\boldsymbol{m}^{e}_{\Pi,\boldsymbol{\Sigma}^s} - \boldsymbol{b}^{\{0,s\}}_{\Pi}\boldsymbol{m}^{i}_{\Pi,\Sigma^s} + \boldsymbol{b}^{\{l\}}_{\Pi}(\boldsymbol{I}+\boldsymbol{d}^{\{l\},\{l\}}_{\Pi})^{-1}[-\boldsymbol{c}^{\{l\}}_{\Pi},\boldsymbol{d}^{\{l\},\{0,s\}}_{\Pi}]\begin{bmatrix} m^{e}_{\Pi,\boldsymbol{\Sigma}^s} \\ m^{i}_{\Pi,\boldsymbol{\Sigma}^s}\end{bmatrix} \\
&\quad - \boldsymbol{b}^{\{l\}}_{\boldsymbol{\sigma}}(\boldsymbol{I}+\boldsymbol{d}^{\{l\},\{l\}}_{\boldsymbol{\sigma}})^{-1}\boldsymbol{u}^{\{l\},i}_{\Pi} + \boldsymbol{u}^{e}_{\Pi} \\
\boldsymbol{F}_{\boldsymbol{\sigma}}(\boldsymbol{u}) &= \boldsymbol{c}^{\{0,s\}}_{\Pi}\boldsymbol{m}^{e}_{\Pi,\boldsymbol{\Sigma}^s} - \boldsymbol{d}^{\{0,s\},\{0,s\}}_{\Pi}\boldsymbol{m}^{i}_{\Pi,\boldsymbol{\Sigma}^s} + \boldsymbol{d}^{\{0,s\},\{l\}}_{\Pi}(\boldsymbol{I}+\boldsymbol{d}^{\{l\},\{l\}}_{\Pi})^{-1}[-\boldsymbol{c}^{\{l\}}_{\Pi},\boldsymbol{d}^{\{l\},\{0,s\}}_{\Pi}]\begin{bmatrix} m^{e}_{\Pi,\boldsymbol{\Sigma}^s} \\ m^{i}_{\Pi,\boldsymbol{\Sigma}^s}\end{bmatrix} \\
&\quad - \boldsymbol{d}^{\{0,s\},\{l\}}_{\Pi}(\boldsymbol{I}+\boldsymbol{d}^{\{l\},\{l\}}_{\Pi})^{-1}\boldsymbol{u}^{\{l\},i}_{\Pi} + \boldsymbol{u}^{\{0,s\},i}_{\Pi} \\
\boldsymbol{G}_{\boldsymbol{\sigma}}(\boldsymbol{u}) &= \boldsymbol{c}^{\{l\}}_{\Pi}\boldsymbol{m}^{e}_{\Pi,\boldsymbol{\Sigma}^s} - \boldsymbol{d}^{\{l\},\{0,s\}}_{\Pi}\boldsymbol{m}^{i}_{\Pi,\boldsymbol{\Sigma}^s} + \boldsymbol{d}^{\{l\},\{l\}}_{\Pi}(\boldsymbol{I}+\boldsymbol{d}^{\{l\},\{l\}}_{\Pi})^{-1}[-\boldsymbol{c}^{\{l\}}_{\Pi},\boldsymbol{d}^{\{l\},\{0,s\}}_{\Pi}]\begin{bmatrix} m^{e}_{\Pi,\boldsymbol{\Sigma}^s} \\ m^{i}_{\Pi,\boldsymbol{\Sigma}^s}\end{bmatrix} \\
&\quad - \boldsymbol{d}^{\{l\},\{l\}}_{\Pi}(\boldsymbol{I}+\boldsymbol{d}^{\{l\},\{l\}}_{\Pi})^{-1}\boldsymbol{u}^{\{l\},i}_{\Pi} + \boldsymbol{u}^{\{l\},i}_{\Pi}
\end{aligned}
\tag{4.3}
$$

Proof. The proof consists of two steps: first, we determine the list of $\Omega_{\boldsymbol{\sigma}}$ that can stable and then we ensure that they do not contain their equilibrium candidates if (4.1) holds.
Step 1. It is possible to decompose the switching regions in two groups: One where there is is at least one excitatory node in linear equilibrium and the other one where all excitatory nodes are on positive or negative saturation.
Let $\boldsymbol{\Pi}'_{\boldsymbol{\sigma}}$ the permutation matrix such that for a concrete region $\boldsymbol{\sigma}$ matches all the excitatory nodes in linear state and all the inhibitory ones to the right bottom elements of the system matrix, so $\boldsymbol{\Pi}'\boldsymbol{\sigma} = [\mathbf{0}_{n-r}, \underbrace{l,\ldots,l}_{\text{r}}, \sigma_{n+1},\ldots,\sigma_{n+m}]^T$.
The coefficient matrix $-\boldsymbol{I}+\boldsymbol{\Sigma}^l\boldsymbol{W}$ in the region $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ then satisfies:

$$
\boldsymbol{\Pi}'_{\boldsymbol{\sigma}}(-\boldsymbol{I}+\boldsymbol{\Sigma}^l\boldsymbol{W})\boldsymbol{\Pi}'^{T}_{\boldsymbol{\sigma}} = \begin{bmatrix} -\boldsymbol{I}_{n-r} & 0 \\ * & P \end{bmatrix}
$$

where:

$$
P = \begin{bmatrix} -\boldsymbol{I}_r - \boldsymbol{a}_l & * \\ * & -\boldsymbol{I}_m + \boldsymbol{\Sigma}^l_{2,2}\boldsymbol{d} \end{bmatrix}
$$

With that, the eigenvalues of $-\boldsymbol{I}+\boldsymbol{\Sigma}^l\boldsymbol{W}$ consists of (-1) with multiplicity $n-r$ and then, the eigenvalues of $\boldsymbol{P}$. Therefore,

- If $r>0$, then $\Omega_{\boldsymbol{\sigma}}$ is unstable because of assumption (2).
- If $r=0$ then we cannot guarantee whether the equilibrium candidate will be stable or unstable [1], so we are going to check that the equilibrium candidate does not fall into its region if (4.1) holds.

[1]If P was totally Hurwitz stable equilibria could be guaranteed for every switching region and the theorem would become an if and only if statement.

Step 2. Let $\boldsymbol{\sigma} = [\boldsymbol{\sigma}', \boldsymbol{\sigma}'']^T$ where $\boldsymbol{\sigma}' = \{0, s\}^n$, as $r = 0$, and $\boldsymbol{\sigma}'' = \{0, s, l\}^m$. Let T set of linear inhibitory nodes in the region, and let $\boldsymbol{\Pi_\sigma}$ be the permutation matrix that applied to $\boldsymbol{\sigma}$, matches every linear inhibiroty node to the bottom right elements of the system matrix ($\boldsymbol{\Pi_\sigma \sigma} = [\mathbf{0}_n, \mathbf{0}_{m-t}, \underbrace{l, \ldots, l}_{t}]$).

Taking that into account, the equilibria points of the permuted system, are the following:

$$\boldsymbol{x}^*_\Pi = \boldsymbol{\Pi_\sigma}(\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})^{-1} \boldsymbol{\Pi}^T_{\boldsymbol{\sigma}}((\boldsymbol{\Pi_\sigma \Sigma}^l \boldsymbol{u} + \boldsymbol{\Pi_\sigma \Sigma}^s \boldsymbol{m})$$

So using the same notation introduced on previous sections the expression can be developed:

$$\boldsymbol{x}^*_\Pi = \left(\begin{bmatrix} \boldsymbol{I}_n & 0 & 0 \\ 0 & \boldsymbol{I}_{m-t} & 0 \\ 0 & 0 & \boldsymbol{I}_t \end{bmatrix} - \boldsymbol{\Pi_\sigma \Sigma}^l \boldsymbol{\Pi}^T_{\boldsymbol{\sigma}} \begin{bmatrix} \boldsymbol{a} & -\boldsymbol{b}_\Pi^{\{0,s\}} & -\boldsymbol{b}_\Pi^{\{l\}} \\ \boldsymbol{c}_\Pi^{\{0,s\}} & -\boldsymbol{d}_\Pi^{\{0,s\},\{0,s\}} & -\boldsymbol{d}_\Pi^{\{0,s\},\{l\}} \\ \boldsymbol{c}_\Pi^{\{l\}} & -\boldsymbol{d}_\Pi^{\{l\},\{0,s\}} & -\boldsymbol{d}_\Pi^{\{l\},\{l\}} \end{bmatrix} \right)^{-1} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ \boldsymbol{u}_\Pi^{\{l\},i} \end{bmatrix} + \begin{bmatrix} m^e_{\Pi,\boldsymbol{\Sigma}^s} \\ m^i_{\Pi,\boldsymbol{\Sigma}^s} \\ 0 \end{bmatrix} \right)$$

$$\boldsymbol{x}^*_\Pi = \left(\begin{bmatrix} \boldsymbol{I}_n & 0 & 0 \\ 0 & \boldsymbol{I}_{m-t} & 0 \\ \boldsymbol{c}_\Pi^{\{l\}} & -\boldsymbol{d}_\Pi^{\{l\},\{0,s\}} & -\boldsymbol{d}_\Pi^{\{l\},\{l\}} \end{bmatrix} \right)^{-1} \cdot \left(\begin{bmatrix} m^e_{\Pi,\boldsymbol{\Sigma}^s} \\ m^i_{\Pi,\boldsymbol{\Sigma}^s} \\ \boldsymbol{u}_\Pi^{\{l\},i} \end{bmatrix} \right)$$

Considering the inverse 2×2 block matrix:

$$\boldsymbol{x}^*_\Pi = \begin{bmatrix} \boldsymbol{I}_{n+m-t} & 0 \\ (\boldsymbol{I}_t + \boldsymbol{d}^{l,l}_{\boldsymbol{\sigma}})^{-1} \begin{bmatrix} -\boldsymbol{c}^l_{\boldsymbol{\sigma}} & \boldsymbol{d}^{l,0}_{\boldsymbol{\sigma}} \end{bmatrix} & (\boldsymbol{I}_t + \boldsymbol{d}^{l,l}_{\boldsymbol{\sigma}})^{-1} \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} m^e_{\Pi,\boldsymbol{\Sigma}^s} \\ m^i_{\Pi,\boldsymbol{\Sigma}^s} \end{bmatrix} \\ \boldsymbol{u}_\Pi^{\{l\},i} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} m^e_{\Pi,\boldsymbol{\Sigma}^s} \\ m^i_{\Pi,\boldsymbol{\Sigma}^s} \end{bmatrix} \\ (\boldsymbol{I}_t + \boldsymbol{d}_\Pi^{\{l\},\{l\}})^{-1} \begin{bmatrix} -\boldsymbol{c}_\Pi^{\{l\}} & \boldsymbol{d}_\Pi^{\{l\},\{0,s\}} \end{bmatrix} \begin{bmatrix} m^e_{\Pi,\boldsymbol{\Sigma}^s} \\ m^i_{\Pi,\boldsymbol{\Sigma}^s} \end{bmatrix} + (\boldsymbol{I}_t + \boldsymbol{d}_\Pi^{\{l\},\{l\}})^{-1} \boldsymbol{u}_\Pi^{\{l\},i} \end{bmatrix}$$

So it is straightforward to verify that:

$$\boldsymbol{\Pi_\sigma W \Pi}^t_{\boldsymbol{\sigma}} \boldsymbol{x}^*_\Pi + \boldsymbol{\Pi_\sigma u} = \begin{bmatrix} \boldsymbol{H_\sigma} \\ \boldsymbol{F_\sigma} \\ \boldsymbol{G_\sigma} \end{bmatrix} \tag{4.4}$$

Where $\boldsymbol{H_\sigma}$, $\boldsymbol{F_\sigma}$, $\boldsymbol{G_\sigma}$ are defined as in the theorem statement. Then, $\boldsymbol{\Pi_\sigma W \Pi}^t_{\boldsymbol{\sigma}} \boldsymbol{x}^*_{\boldsymbol{\sigma}} + \boldsymbol{\Pi_\sigma u} \in \Omega_{\boldsymbol{\sigma}}$ if and only if $\boldsymbol{u} \in \bigcap_{i=1}^N y_{\boldsymbol{\sigma}', \boldsymbol{\sigma}'', i}$. Therefore, for no region to contain its equilibrium candidate it is sufficient that:

$$\boldsymbol{u} \in \mathbb{R}^N \setminus Y \tag{4.5}$$

□

Remark 4.1.2. ***(Necessary and sufficient conditions).***
Consider the dynamics (2.7)-(2.8) and assume $\boldsymbol{W}$ verifies assumption (2). Then, the network does not have any stable equilibria if and only if

$$\boldsymbol{u} \in \mathbb{R}^N \setminus Z, \tag{4.6}$$

where $Z \subseteq Y$ is

$$Z = \bigcup_{\boldsymbol{\sigma}' \in \{0,s\}^n} \bigcup_{\substack{\boldsymbol{\sigma}'' \in \{0,s,l\}^m \\ \boldsymbol{\sigma}'' unstable}} \left(\bigcap_{i=1}^{N} y_{\boldsymbol{\sigma}', \boldsymbol{\sigma}'', i} \right)$$

4.2 Complementary approach

With the aim of proposing sufficient conditions for the system to oscillate, a further study of the the region region $\mathbb{R}^N \backslash Y$ and how we model $\boldsymbol{u}$ such that $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$ is convenient.
First of all, $\mathbb{R}^N \backslash Y$ will be rewritten as Y^C. If the dynamics (2.7)-(2.8) are considered without permuting the matrices, then, for a given $\boldsymbol{\sigma}$, the equilibrium point is the following:

$$\boldsymbol{x}^*_{\boldsymbol{\sigma}} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})^{-1} (\boldsymbol{\Sigma}^l \boldsymbol{u} + \boldsymbol{\Sigma}^s \boldsymbol{m})$$

Defining $\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}) \triangleq \boldsymbol{W}\boldsymbol{x}^*_{\boldsymbol{\sigma}} + \boldsymbol{u} = \boldsymbol{W}(\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})^{-1} (\boldsymbol{\Sigma}^l \boldsymbol{u} + \boldsymbol{\Sigma}^s \boldsymbol{m}) + \boldsymbol{u}$, leads to a $Y = \bigcup_{\boldsymbol{\sigma}} \cap_{i=1}^{N} \boldsymbol{y}_{\boldsymbol{\sigma},i}$ then $\boldsymbol{y}_{\boldsymbol{\sigma},i}$ can be expressed as:

$$\boldsymbol{y}_{\boldsymbol{\sigma},i} = \begin{cases} \boldsymbol{u} \mid (\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \geq m_i & \forall i \ s.t. \ \sigma_i = s \\ \boldsymbol{u} \mid (\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \leq 0 & \forall i \ s.t. \ \sigma_i = 0 \\ \boldsymbol{u} \mid (\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i \leq m_i \ \& \ \geq 0 & \forall i \ s.t. \ \sigma_i = l \end{cases}$$

This region in terms of Y is equivalent to:

$$Y^C = \left(\bigcup_{\boldsymbol{\sigma}} \bigcap_{i=1}^{N} y_{\boldsymbol{\sigma},i} \right)^C = \bigcap_{\boldsymbol{\sigma}} \left(\bigcap_{i=1}^{N} y_{\boldsymbol{\sigma},i} \right)^C = \bigcap_{\boldsymbol{\sigma}} \bigcup_{i=1}^{N} y^C_{\boldsymbol{\sigma},i}$$

Where

$$\bigcup_{i=1}^{N} y^C_{\boldsymbol{\sigma},i} = \begin{cases} \boldsymbol{u} \mid (\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i < m_i & for \ some \ \ i \ s.t. \ \sigma_i = s \\ \boldsymbol{u} \mid (\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i > 0 & for \ some \ \ i \ s.t. \ \sigma_i = 0 \\ \boldsymbol{u} \mid (\boldsymbol{J}_{\boldsymbol{\sigma}}(\boldsymbol{u}))_i > m_i \ or \ < 0 & for \ some \ i \ s.t. \ \sigma_i = l \end{cases}$$

This expression of Y^C will be used now on for its simplicity, as now for each switching region in order for the equilibrium candidate to fall outside only one of the inequalities needs to be verified.

4.3 Inequalities for an arbitrary network

With the expression of Y^C we can now provide the different union sets of inequalities that will have to be verified in each different kind of switching region:

- $\boldsymbol{\sigma} = (0, 0, \ldots, 0)$
  The union set of inequalities that must be verified for this $\boldsymbol{\sigma}$ are the following:

$$\Big\{ u_t > 0 \quad \text{For some } t \in \{1, \ldots, m+n\} \tag{4.7}$$

- $\boldsymbol{\sigma} = (\Sigma^{s,e}, \Sigma^{0,e}, \Sigma^{0,i}) \qquad s.t \mid \Sigma^{s,e} \mid + \mid \Sigma^{0,e} \mid = n$ and $\mid \Sigma^{s,e} \mid > 0$ and $\mid \Sigma^{0,i} \mid = m$
  The union set of inequalities for $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$ is:

$$\begin{cases} u_e < m_e - \sum_{t\in\{1,\ldots,n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t & for\ e \in \{1, \ldots, n\}\ and\ (\Sigma^s)_{e,e} = 1 \\ u_e > - \sum_{t\in\{1,\ldots,n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t & for\ e \in \{1, \ldots, n\}\ and\ (\Sigma^s)_{e,e} = 0 \\ u_i > - \sum_{t\in\{1,\ldots,n\}} (\Sigma^s)_{t,t} |W|_{i,t} m_t & for\ i \in \{n+1, \ldots, m+n\} \\ & and\ (\Sigma^s)_{i,i} = 0 \end{cases} \tag{4.8}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{0,i}, \Sigma^{s,i}) \qquad s.t \mid \Sigma^{0,i} \mid + \mid \Sigma^{s,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{0,e} \mid = n$
  The union set of inequalities for the equilibrium point not to fall into the region $\boldsymbol{x}^* \in \mathbb{R}^N \backslash \Omega_{\boldsymbol{\sigma}}$ has the following form:

$$\begin{cases} u_e > \sum_{t\in\{n+1,\ldots,m+n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t & for\ e \in \{1, \ldots, n\} \\ & and\ (\Sigma^s)_{e,e} = 0 \\ u_i < m_i + \sum_{t\in\{n+1,\ldots,m+n\}} \big((\Sigma^s)_{t,t} |W|_{i,t} m_t \big) & for\ i \in \{n+1, \ldots, m+n\} \\ & and\ (\Sigma^s)_{i,i} = 1 \\ u_i > \sum_{t\in\{n+1,\ldots,m+n\}} \big((\Sigma^s)_{t,t} |W|_{i,t} m_t \big) & for\ i \in \{n+1, \ldots, m+n\} \\ & and\ (\Sigma^s)_{kk} = 0 \end{cases} \tag{4.9}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{0,i}, \Sigma^{s,i}) \qquad s.t \mid \Sigma^{0,e} \mid + \mid \Sigma^{s,e} \mid = n$ and $s.t \mid \Sigma^{0,i} \mid + \mid \Sigma^{s,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{s,e} \mid > 0$
  The union set of inequalities, for this $\boldsymbol{\sigma}$ most general regions where no node is linear state, is:

$$\begin{cases} u_e < m_e - \sum_{t\in\{1,\ldots,n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t + \sum_{t\in\{n+1,\ldots,m+n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t \\ \qquad for\ e \in \{1, \ldots, n\}\ and\ (\Sigma^s)_{e,e} = 1 \\ \\ u_e > - \sum_{t\in\{1,\ldots,n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t + \sum_{t\in\{n+1,\ldots,m+n\}} (\Sigma^s)_{t,t} |W|_{e,t} m_t \\ \qquad for\ e \in \{1, \ldots, n\}\ and\ (\Sigma^s)_{e,e} = 0 \end{cases}$$

$$\begin{cases} u_i < m_i - \sum_{t\in\{1,\dots,n\}}(\Sigma^s)_{t,t}|W|_{i,t}m_t + \sum_{t\in\{2,\dots,m+1\}}\big((\Sigma^s)_{t,t}|W|_{i,t}m_t\big) \\ \qquad\qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^s)_{i,i} = 1 \\ \\ u_i > -\sum_{t\in\{1,\dots,n\}}(\Sigma^s)_{t,t}|W|_{i,t}m_t + \sum_{t\in\{n+1,\dots,m+n\}}\big((\Sigma^s)_{t,t}|W|_{i,t}m_t\big) \\ \qquad\qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^s)_{i,i} = 0 \end{cases} \tag{4.10}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$ and that $\mid \Sigma^{0,e} \mid + \mid \Sigma^{s,e} \mid = n$
  The generalized inequalities that describes the valid inputs for these regions are:

$$\begin{cases} u_e > \sum\limits_{t\in 1,\dots,m+n}\big(-sg(|W|_{e,t})|W|_{e,t}(\sum\limits_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) \\ \qquad for\ e \in \{1,\dots,n\}\ and\ (\Sigma^s)_{e,e} = 0 & (in.\ 0) \\ \\ u_e < \sum\limits_{t\in 1,\dots,m+n}\big(-sg(|W|_{e,t})|W|_{e,t}(\sum\limits_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + m_e \\ \qquad for\ e \in \{1,\dots,n\}\ and\ (\Sigma^s)_{e,e} = 1 & (in.\ 1) \\ \\ \sum\limits_{t\in 1,\dots,m+n}\Big(sg(W_{i,t})|W|_{i,t}(\sum\limits_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\Big) + u_i < 0 \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{i,i} = 1 & (in.\ 2) \\ \\ \sum\limits_{t\in 1,\dots,m+n}\Big(sg(W_{i,t})|W|_{i,t}(\sum\limits_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\Big) + u_i > m_i \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{i,i} = 1 & (in.\ 3) \\ \\ \sum\limits_{t\in 1,\dots,m+n}\big(sg(W_{i,t})|W|_{i,t}(\sum\limits_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + u_i < m_i \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{ii} = 0\ and\ (\Sigma^s)_{ii} = 1 & (in.\ 4) \\ \\ \sum\limits_{t\in 1,\dots,m+1}\big(sg(W_{i,t})|W|_{i,t}(\sum\limits_{r\in 1,\dots,m+1}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + u_i > 0 \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{ii} = 0\ and\ (\Sigma^s)_{ii} = 0 & (in.\ 5) \end{cases}$$

$$\tag{4.11}$$

Chapter 5

Geometrical approach to the linear inequalities

The following section will present and proof several results that enable the characterization, in a geometric way, of the inequalities and the regions they enclose.
The goal will be to understand which are the restrictions on $\boldsymbol{u}$ and how can we shape it in order to verify the inequalities.
The results discussed in the following section will mainly be regarding the switching zones where inhibitory nodes in linear state are to be found, as we are assuming that the switching regions involving one excitatory node in linear state are always going to be unstable (2).

Lemma 5.0.1. *For $\boldsymbol{x}^*_{\boldsymbol{\sigma}} \notin \boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ where $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ is such that at least one inhibitory node is in linear mode while all the excitatory are in saturation, it is necessary and sufficient that, for every switching region, at least one of the following inequalities is held:*

$$\begin{cases} u_e > \sum\limits_{t\in 1,\dots,m+n} \big(- sg(|W|_{e,t})|W|_{e,t}(\sum\limits_{r\in 1,\dots,m+n} (K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) \\ \qquad\qquad for\ e \in \{1,\dots,n\}\ and\ (\Sigma^s)_{e,e} = 0 & (in.\ 0) \\ \\ u_e < \sum\limits_{t\in 1,\dots,m+n} \big(- sg(|W|_{e,t})|W|_{e,t}(\sum\limits_{r\in 1,\dots,m+n} (K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + m_e \\ \qquad\qquad for\ e \in \{1,\dots,n\}\ and\ (\Sigma^s)_{e,e} = 1 & (in.\ 1) \\ \\ \sum\limits_{t\in 1,\dots,m+n} \Big(sg(W_{i,t})|W|_{i,t}(\sum\limits_{r\in 1,\dots,m+n} (K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\Big) + u_i < 0 \\ \qquad\qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{i,i} = 1 & (in.\ 2) \end{cases}$$

$$\begin{cases} \sum_{t\in 1,\dots,m+n} \Big(sg(W_{i,t})|W|_{i,t}(\sum_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\Big) + u_i > m_i \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{i,i} = 1 \qquad\qquad (in.\ 3) \\ \\ \sum_{t\in 1,\dots,m+n} \big(sg(W_{i,t})|W|_{i,t}(\sum_{r\in 1,\dots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + u_i < m_i \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{ii} = 0\ and\ (\Sigma^s)_{ii} = 1 \qquad\qquad (in.\ 4) \\ \\ \sum_{t\in 1,\dots,m+n} \big(sg(W_{i,t})|W|_{i,t}(\sum_{r\in 1,\dots,m+1}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + u_i > 0 \\ \qquad for\ i \in \{n+1,\dots,m+n\}\ and\ (\Sigma^l)_{ii} = 0\ and\ (\Sigma^s)_{ii} = 0 \qquad\qquad (in.\ 5) \end{cases} \tag{5.1}$$

Notation 5.0.2. *Each region has $n + m$ inequalities that can be verified. For each switching region, we are going to refer to those inequalities steering form nodes in linear state as **linear inequalities** ((in. 2) and (in. 3)), while for the nodes in either positive or negative saturation ((in. 0), (in. 1),(in. 4) and (in. 5)) we are going to use the term **saturated inequalities**.*

Lemma 5.0.3. ***(Equivalence of the linear inequalities).***
For every i linear node, the LHS of the linear inequalities is equivalent to:

$$\sum_{t\in 1,\dots,m+n} \Big(sg(W_{i,t})|W|_{i,t}(\sum_r(K^l_{i,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\Big) + u_i \equiv \sum_{t\in 1,\dots,m+n} K^l_{i,t}(\Sigma^l)_{tt}u_t + K^l_{i,t}(\Sigma^s)_{tt}m_t \tag{5.2}$$

Proof. As stated before, the equilibrium points of the system can be described as $\boldsymbol{x}^*_{\boldsymbol{\sigma}} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W})^{-1}(\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{u} + \boldsymbol{\Sigma}^s_{\boldsymbol{\sigma}}\boldsymbol{m})$. Substituting this expression in the dynamics of the system, and using the definition of equilibrium point, one gets that:

$$\boldsymbol{W}\boldsymbol{x}^*_{\boldsymbol{\sigma}} + \boldsymbol{u} = \boldsymbol{x}^*_{\boldsymbol{\sigma}}$$
$$\boldsymbol{W}(\boldsymbol{I} - \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W})^{-1}(\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{u} + \boldsymbol{\Sigma}^s_{\boldsymbol{\sigma}}\boldsymbol{m}) + \boldsymbol{u} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W})^{-1}(\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{u} + \boldsymbol{\Sigma}^s_{\boldsymbol{\sigma}}\boldsymbol{m})$$

So in particular, taking into account the rows corresponding to nodes in linear state, its straightforward to verify that the equivalence holds.

□

Lemma 5.0.4. ***(Bijection of the invalid region).***
Let's consider a fixed $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ and with it its corresponding $\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}$ and $\boldsymbol{\Sigma}^s_{\boldsymbol{\sigma}}$.
Let L be a set of nodes such that $L = \{i \in \{n+1,\dots,m+n\} \mid (\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}})_{i,i} = 1\}$, in other words, the set of inhibitory nodes in linear state.

Let S be a set such that $S = \{i \in \{1, \dots, n+m\} \mid (\boldsymbol{\Sigma}^s_{\boldsymbol{\sigma}})_{i,i} = 1\}$, in other words, the set of nodes, either inhibitory or excitatory, in positive saturation state.
Let $\mathbb{V} = V_1 \times V_2 \times \cdots \times V_{m+n}$ be a manifold of the $\mathbb{R}^{m+n}$ euclidean where V_i is:

$$V_i = \begin{cases} \mathbb{R} & \textit{If } (\Sigma^l)_{i,i} = 1 \\ \{m_i\} & \textit{If } (\Sigma^l)_{i,i} = 0 \textit{ and } (\Sigma^s)_{i,i} = 1 \\ \{0\} & \textit{Otherwise} \end{cases}$$

Now, let K be the lineal application defined as $K \triangleq (\boldsymbol{I} - \boldsymbol{\Sigma}^l\boldsymbol{W})^{-1}$ such that:

$$\begin{aligned} K \colon \mathbb{V} &\longrightarrow \mathbb{V} \\ \boldsymbol{v} &\longmapsto \boldsymbol{K}(\boldsymbol{v}) \end{aligned}$$

Let $\mathbb{U} \subset \mathbb{V}$ be a region described as $\mathbb{U} = U_1 \times U_2 \times \cdots \times U_{m+n}$, where U_i is:

$$U_i = \begin{cases} [0, m_i] & \textit{If } (\Sigma^l)_{i,i} = 1 \\ \{m_i\} & \textit{If } (\Sigma^l)_{i,i} = 0 \textit{ and } (\Sigma^s)_{i,i} = 1 \\ \{0\} & \textit{Otherwise} \end{cases}$$

Let $\boldsymbol{u}$ be the input of the system.
Let $\boldsymbol{v} \triangleq (\boldsymbol{\Sigma}^l\boldsymbol{u} + \boldsymbol{\Sigma}^s\boldsymbol{m}) \in V$.
If $\boldsymbol{v} \notin \boldsymbol{K^{-1}}(\mathbb{U})$ at least one of the inequalities of the form (in. 2) or (in. 3), i.e. the linear inequalities, will be always held.

Proof. We will prove that $\forall \boldsymbol{v} \in \mathbb{V} \backslash \boldsymbol{K^{-1}}(\mathbb{U})$ either (*in.* 1) or (*in.* 2) holds.
First of all, one must check that $\boldsymbol{K}$ is well defined and bijective:

- **Well defined:** As the i-th row equal to identity if and only if $(\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}})_{i,i} = 0$, we can also say that $\forall \boldsymbol{v} \in \mathbb{V}$, $\boldsymbol{K}(\boldsymbol{v}) \in \mathbb{V}$. Then due to the form of $\boldsymbol{K} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W})^{-1}$, and assuming that the matrix admits inverse, as it is a linear application it is straightforward to verify that it is well defined.

- **Bijective:** It is a direct consequence of $\boldsymbol{K}$ being a linear application that admits inverse. In fact, $\boldsymbol{K}^{-1} = \big((\boldsymbol{I} - \boldsymbol{\Sigma}^l\boldsymbol{W})^{-1}\big)^{-1} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l\boldsymbol{W})$.

Now, let $\mathbb{N} \subset \mathbb{V}$, the set of where some inequality of the form (*in.* 1) or (*in.* 2) holds for some $i \in L$. That is:

$$\mathbb{N} = \{\boldsymbol{w} \in \mathbb{V} \mid (w)_i < 0 \textit{ or } (w)_i > m_i \textit{ for some } i \in L\}$$

The set where none linear inequality will be held is:

$$\mathbb{V} \backslash \mathbb{N} = \{\boldsymbol{w} \in \mathbb{V} \mid (w)_i \in [0, m_i] \ \forall i \in L\} = \mathbb{U}$$

As $\boldsymbol{K}$ is a bijection, it is sufficient and necessary that $\boldsymbol{v} \in \mathbb{V} \backslash \boldsymbol{K}^{-1}(U)$ in order for it to verify some of the linear inequalities.

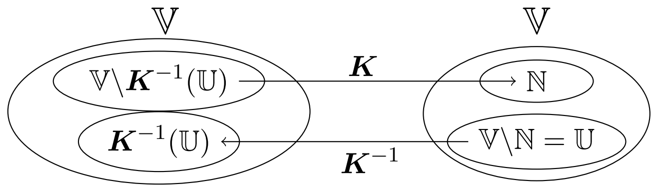


Figure 5.1: Bijection between valid and invalid manifolds

Furthermore, developing the application, one can see that the non-valid region is a polyhedron $\mathbb{P} \subset \mathbb{V}$ described by the lineal application:

$$\boldsymbol{K}^{-1}(\mathbb{U}) = (\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})\mathbb{U} \triangleq \mathbb{P}^L$$

So it is equivalent for some inequalities to be verified that for $\boldsymbol{u} \in \mathbb{R}^N$, with $\boldsymbol{v} = (\boldsymbol{\Sigma}^l \boldsymbol{u} + \boldsymbol{\Sigma}^s \boldsymbol{m}) \in V$, then $\boldsymbol{v} \notin \mathbb{P}^L$.

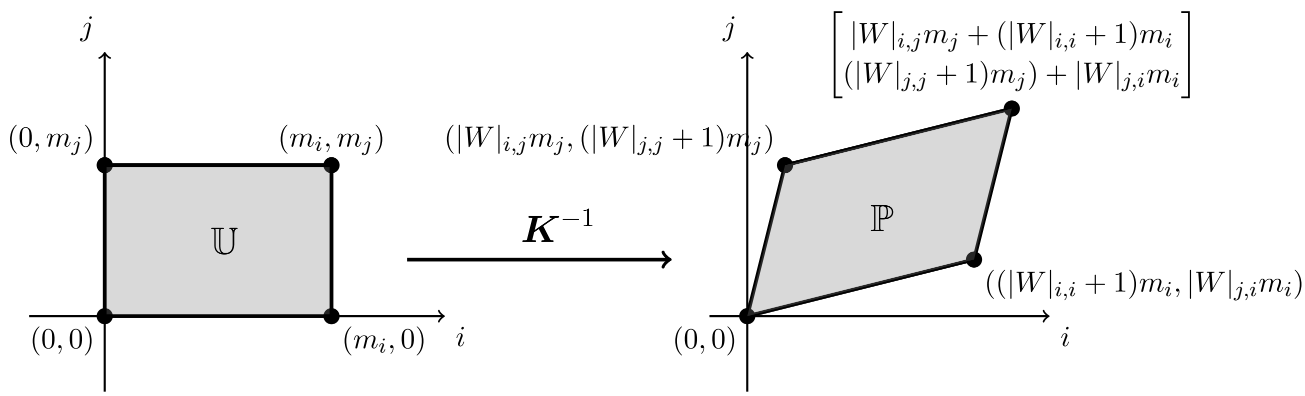


Figure 5.2: Example of $\mathbb{P}^{i,j}$ for an arbitrary i, j. If $\boldsymbol{v} \notin \mathbb{P}$ then $K(v) \notin \mathbb{U}$ and some inequality will be held.

$\square$

5.1 Results for regions with no excitatory node in saturation

The next section will discuss results for switching regions where no excitatory is found in positive saturation. These results will later be generalized in the subsequent section, for any possible set of excitatory nodes in positive saturation. However, the proofs will be presented for the non-generalized for the shake of simplicity.

Theorem 5.1.1. ***(Parallelogram Theorem).***

Let L be a fixed set of inhibitory nodes in linear state, such that $|L| \geq 2$.
Let $\boldsymbol{\Sigma}^l$ be its associated diagonal matrix, where $(\Sigma^l)_{i,i} = 1 \Leftrightarrow i \in L$.
Let $\mathbb{U} = \prod_i (\Sigma^l)_{i,i}[0, m_i]$ and let $\boldsymbol{K} = (\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})^{-1}$.
Let $\mathbb{P}^L = \boldsymbol{K}^{-1}(\mathbb{U}) = (\boldsymbol{I} - \Sigma^l \boldsymbol{W})\mathbb{U} \subset \mathbb{R}^N$.
Let $Q_{i,j}$ be a parallelogram in $\mathbb{R}^2$ defined by the convex combination of the points:

$$Q_{i,j} = \begin{cases} \big(0, 0\big) \\ \big((|W|_{j,j} + 1)m_j, |W|_{i,j} m_j\big) \\ \big((|W|_{j,i} m_i, |W|_{i,i} + 1)m_i\big) \\ \big(|W|_{j,j} + 1)m_j + |W|_{j,i} m_i, |W|_{i,j} m_j + (|W|_{i,i} + 1)m_i)\big) \end{cases} \tag{5.3}$$

Let $t_{i,j} \in L$ be an inhibitory node such that $t_{i,j} \neq i, j$.
For each $t_{i,j}$ let $O_{t_{i,j}}$ be described as:

$$O_{t_{i,j}} = \begin{cases} (O_{t_{i,j}})_k = 0 & \text{If } k \notin L \\ (O_{t_{i,j}})_k = |W|_{k,t_{i,j}} m_{t_{i,j}} & \text{If } k \in L \end{cases} \tag{5.4}$$

Let $\Pi_{i,j}$ be the projection to the i,j plane.
Let $\boldsymbol{u}$ be the input of the system and $\boldsymbol{v} = \boldsymbol{\Sigma}^l \boldsymbol{u}$.
If $\exists\ i, j \in L$ such that $\Pi_{i,j}\big(\boldsymbol{u} - \sum\limits_{\forall t_{i,j}} \Gamma_{t_{i,j}} O_{t_{i,j}}\big) \notin Q_{i,j}\ \forall \Gamma_{t_{i,j}} \in [0, 1]$, then $\boldsymbol{v} \notin \mathbb{P}^L$, in other words $(\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})^{-1}(\boldsymbol{\Sigma}^l \boldsymbol{u}) \notin \mathbb{U}$, so some of the linear inequalities, of the form (in. 2) or (in. 3), will be held.
Furthermore, let $T_{i,j} \subseteq L$ be a set of nodes such that $\{i, j\} \in T$ and let S be the set of positive saturated nodes such that $S \subseteq L \backslash T_{i,j}$.
Let $\mathbb{P}^T + \sum_{s \in S} O_s$ be the polyhedron $\mathbb{P}^T$ translated by $\sum_{s \in S} O_s$. Then, it is also true that $(\Sigma^T \boldsymbol{u} + \Sigma^S \boldsymbol{m}) \notin \mathbb{P}^T + \sum_{s \in S} O_s \subseteq \mathbb{P}^L$.

Remark 5.1.2. ***(Interpretation of the parallelogram theorem).***
Imagine that you have a set of three nodes i, j, k with the objective of finding an input $\boldsymbol{u}$ able to verify all the linear inequalities for the region where all the nodes are in linear.
Using lemma (5.0.4) on the bijection of the invalid zone we have that $\boldsymbol{u}$ will verify the inequalities if $\boldsymbol{u} \notin \mathbb{P}^{i,j,k}$.

The invalid region on the i, j, k *plane will look like the following, a linear transformation on a 3D polyhedron. Let, for example,* $A = O_i$*,* $B = O_j$ *and* $C = O_k$*.*

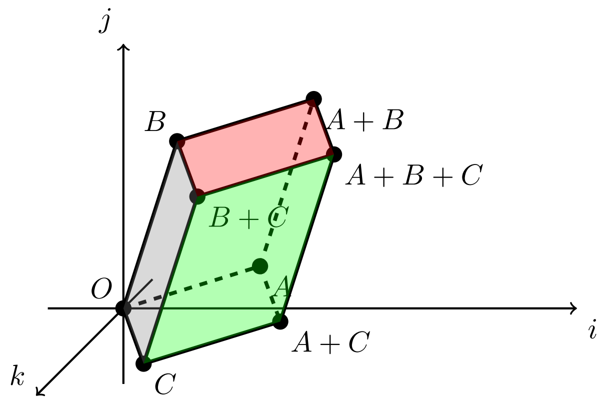


Figure 5.3: Invalid region $\mathbb{P}^{i,j,k}$ represented on the plane i-j-k.

As the theorem does, we care for the projection on a specific 2D plane, in this case the projection used to illustrate the example is the one on the j, k *plane and it will be as the following figure shows.*

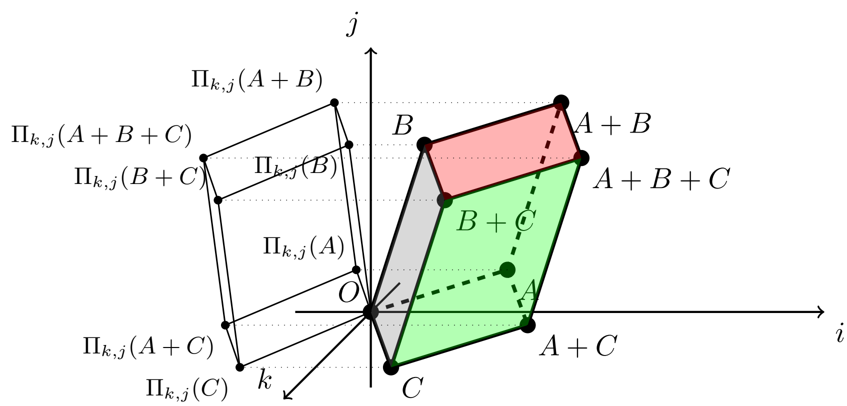


Figure 5.4: Projection on the j-k plane.

The representation into the $k - j$ *plane is as the next figure shows.*
The example, clearly shows that if the input is outside of the parallelogram in two dimensions then it is a sufficient condition for being out of the invalid region outside of $\mathbb{P}^{i,j,k}$*. Furthermore, by construction the input will also be outside* $\mathbb{P}^{i,j} = Q_{i,j}$ *and outside of* $\mathbb{P}^{i,j} + O_A$*.*

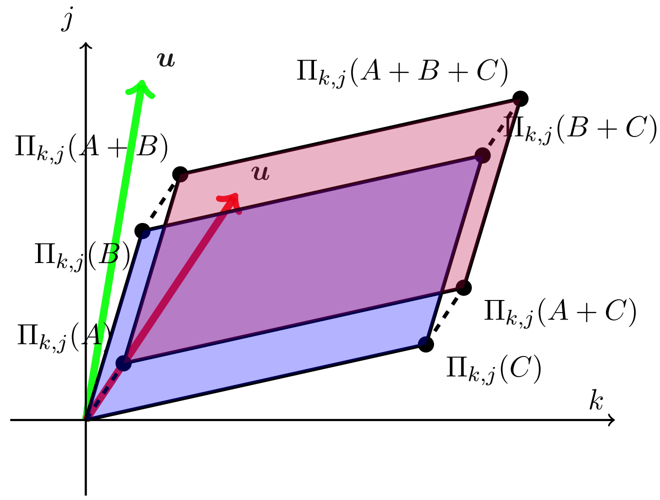


Figure 5.5: Projection on the $k-j$ plane, and representation of a valid and a non-valid input

Proof. We are going to prove the counter positive of the statement.
For a fixed L, if $\boldsymbol{v} \in \mathbb{P}^L$, then $\forall$ pair i,j it always exists a $\boldsymbol{\Gamma} \in [0,1]^{L-2}$ such that $\Pi_{i,j}\big(\boldsymbol{u} - \sum\limits_{\forall t_{i,j}} \Gamma_{t_{i,j}} O_{t_{i,j}}\big) \in \mathbb{Q}_{i,j}$.

First we are going to describe the individual regions where $|L| = 2$ for an arbitrary pair i,j such that $i,j \in L$
The region of non-valid $\boldsymbol{u}$ will be the linear transform by $(I - \Sigma^{i,j}\boldsymbol{W})$ of the polyhedron described by:

$$\mathbb{U}_{i,j} = \boldsymbol{w} \in \mathbb{R}^m \ s.t \ \begin{cases} (w)_k = 0 & If\ k \neq j,i \\ (w)_k \in [0, m_j] & If\ k = j \\ (w)_k \in [0, m_j] & If\ k = i \end{cases}$$

One can see that $(I - \Sigma^{i,j}\boldsymbol{W})\mathbb{U}_{i,j} \equiv Q_{i,j}$.
Plotting the results in the terms in the $u_i - u_j$ plane we have the following non-valid region:

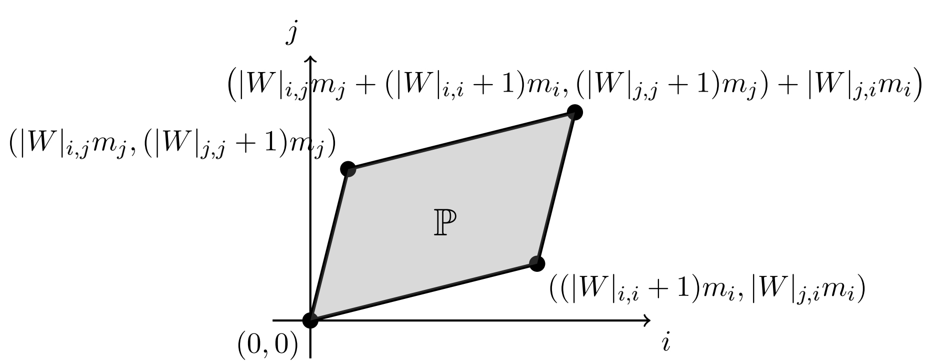


Figure 5.6: Valid region seen as $\mathbb{P}^{i,j} = Q_{i,j}$

Now let's consider $\mathbb{P}^L$, using the same construction but for the L set of nodes.
Let $\boldsymbol{\theta} \in \mathbb{R}^L$ be a vector of indicator functions θ_i and Θ the set of all $\boldsymbol{\theta}$.
With this, an arbitrary point $\boldsymbol{v}'$ laying inside $\mathbb{P}^L$ will be a convex combination of all the vertex and can be written as:

$$(\boldsymbol{v}')_t = \sum_{\boldsymbol{\theta}\in\Theta} \alpha_{\boldsymbol{\theta}} \Big(\big(\sum_{\imath\in L} \theta_i W_{t,i} m_i \big) + \theta_i m_i \Big) \tag{5.5}$$

Such that $\alpha_{\boldsymbol{\theta}} \geq 0$ and $\sum_{\boldsymbol{\theta}\in\Theta} \alpha_{\boldsymbol{\theta}} = 1$.
Reordering the terms, the vector $\boldsymbol{v}$ can be also seen as:

$$\boldsymbol{v}' = \begin{cases} v_i' = \big(\sum_{j\in L} \beta_j |W|_{i,j} m_j \big) + \beta_i m_i & \text{If } i \in L \\ v_i' = 0 & \text{Otherwise} \end{cases} \tag{5.6}$$

Such that $\beta_j \in [0,1]\ \forall j$.
Let's now fix an arbitrary i, j. If $\boldsymbol{\Gamma}$ is such that

$$\boldsymbol{\Gamma} = \Big\{ \Gamma_{t_{i,j}} = \beta_{t_{i,j}} \quad \forall t_{i,j} \in L \backslash \{i,j\} \tag{5.7}$$

then is straightforward to verify that $\Pi_{i,j}(\boldsymbol{u} - \sum_{t_{i,j}} \Gamma_{t_{i,j}} O_{t_{i,j}}) \in Q_{i,j}$, because:

$$\begin{aligned} \Pi_{i,j}(\boldsymbol{u} - \sum_{\forall t_{i,j}} \Gamma_{t_{i,j}} O_{t_{i,j}}) &= \begin{bmatrix} \big(\sum_{t\in L} \beta_j |W|_{i,t} m_t \big) + \beta_i m_i - (\sum_{\forall t_{i,j}}) \Gamma_{t_{i,j}} O_{t_{i,j}}))_i \\ \big(\sum_{t\in L} \beta_j |W|_{j,t} m_t \big) + \beta_j m_j - (\sum_{\forall t_{i,j}}) \Gamma_{t_{i,j}} O_{t_{i,j}}))_j \end{bmatrix} = \\ &= \begin{bmatrix} \big(\sum_{t\in L} \beta_j |W|_{i,t} m_t \big) + \beta_i m_i - (\sum_{\forall t_{i,j}}) \beta_{t_{i,j}} |W|_{i,t_{i,j}} m_{t_{i,j}})_i \\ \big(\sum_{t\in L} \beta_j |W|_{j,t} m_t \big) + \beta_j m_j - (\sum_{\forall t_{i,j}}) \beta_{t_{i,j}} |W|_{j,t_{i,j}} m_{t_{i,j}})_j \end{bmatrix} = \\ &= \begin{bmatrix} \beta_j |W|_{i,j} m_j + \beta_i (|W|_{i,i} + 1) m_i \\ \beta_j (|W|_{j,j} + 1) m_j + \beta_i |W|_{j,i} m_i \end{bmatrix} \in \mathbb{Q}_{i,j} \end{aligned} \tag{5.8}$$

Note that as $\boldsymbol{v} \notin \mathbb{P}^L$, then $\forall T \subseteq L$ such that $i, j \in T$, $\boldsymbol{\Sigma^T u} \notin \mathbb{P}^T$.
Now consider a set S of positive saturated nodes such that $S \subseteq L \backslash \{i,j\}$ and let $T = L \backslash S$. Consider $\boldsymbol{v} = (\Sigma^T \boldsymbol{u} + \Sigma^S \boldsymbol{m})$. We want to show that $\boldsymbol{v} \notin \mathbb{P}^T + \sum_{s\in S} O_s \subseteq \mathbb{P}^L$.
That $\boldsymbol{v} \notin \mathbb{P}^L$ is trivial because $\Pi_{i,j}(\boldsymbol{u}) = \Pi_{i,j}(\boldsymbol{v})$. Now let's see that $P^T + \sum_{s\in S} O_s \subseteq \mathbb{P}^L$.
We have that $P^T + \sum_{s\in S} O_s = \boldsymbol{K}^{-1}(\mathbb{U}')$ where $\mathbb{U}' = U_1' \times \cdots \times U_{n+m}'$ where U_i' is:

$$U_i = \begin{cases} [0, m_i] & \text{If } (\Sigma^T)_{i,i} = 1 \\ \{m_i\} & \text{If } (\Sigma^T)_{i,i} = 0 \text{ and } (\Sigma^S)_{i,i} = 1 \\ \{0\} & \text{Otherwise} \end{cases} \tag{5.9}$$

And $P^L = \boldsymbol{K}^{-1}(\mathbb{U})$ where $\mathbb{U} = U_1 \times \cdots \times U_{n+m}$ where U_i is:

$$U_i = \begin{cases} [0, m_i] & \text{If } (\Sigma^T)_{i,i} = 1 \\ \{m_i\} & \text{If } (\Sigma^T)_{i,i} = 0 \text{ and } (\Sigma^S)_{i,i} = 1 \\ \{0\} & \text{Otherwise} \end{cases} \tag{5.10}$$

As $T \subseteq L$ then $\mathbb{U}' \subseteq \mathbb{U}$ and so $P^T + \sum_{s\in S} O_s = \boldsymbol{K}^{-1}(\mathbb{U}') \subseteq \boldsymbol{K}^{-1}(\mathbb{U}) = P^L$ □

Corollary 5.1.3. ***(Cone approach).***
Let L be a set of inhibitory nodes such that $|L| \geq 2$.
Let $T \subseteq L$ such that $|T| \geq 2$.
Let $\mathbb{S}_{i,j}$, with $i, j \in T \subseteq L$ be the convex cone defined by the 2-dimensional vectors:

$$\mathbb{S}_{i,j} = \bigcup_{\{t\in L\}} \{\boldsymbol{x_{i,j,t}}\} \quad \textit{where} \quad \boldsymbol{x_{i,j,t}} = \begin{cases} (\boldsymbol{x_{i,j,t}})_i = (\boldsymbol{I} - \boldsymbol{W})_{i,t} m_t \\ (\boldsymbol{x_{i,j,t}})_j = (\boldsymbol{I} - \boldsymbol{W})_{j,t} m_t \end{cases} \tag{5.11}$$

Let $\Pi_{i,j}$ be the projection to the i,j plane. If there $\exists i, j$ such that $\Pi_{i,j}(\boldsymbol{u}) \notin \mathbb{S}_{i,j}$ then $(\Sigma^T \boldsymbol{u}) \notin \mathbb{P}^T$ for every $T \subseteq L$ such that $i, j \in T$ and $|T| \geq 2$.
Furthermore, let S be the set of positive saturated nodes such that $S \subseteq L \backslash T$. Let $\mathbb{P}^T + \sum_{s\in S}$ be the polyhedron $\mathbb{P}^T$ translated by $O_S = \sum_{s\in S} O_s$. Then, it is also true that $(\Sigma^T \boldsymbol{u} + \Sigma^S \boldsymbol{m}) \notin \mathbb{P}^T + O_S$.

Remark 5.1.4. ***(Interpretation of the cone approach).***
As the following figure exemplifies, if the input vector $\mathbf{u}$ projected to the $i - j$ plane is outside the cone spanned by $\mathbb{S}_{i,j}$ it will be outside $\mathbb{P}^{i,j}$, outside $\mathbb{P}^{i,j} + O_S$ and outside $\mathbb{P}^{i,j,k}$

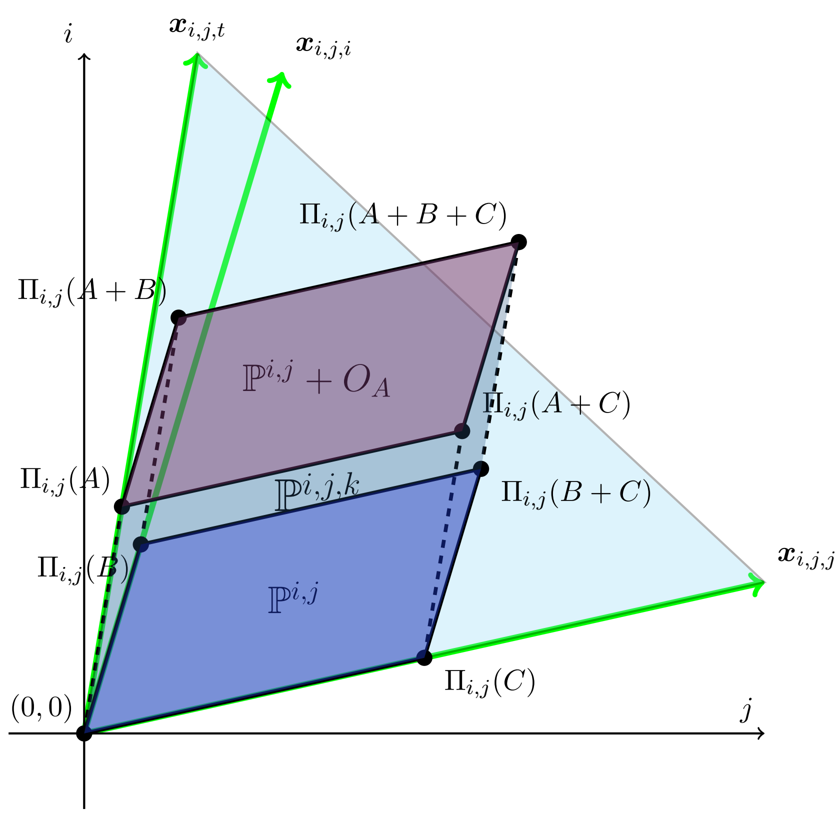


Figure 5.7: Graphical representation of the cone approach

Lemma 5.1.5. ***(Consequences of the cone approach for*** $|L| = 1$***).***
Let w, j a pair of nodes.
Let S be a set of inhibitory nodes in positive saturation, not containing i and j. $S \subseteq I \backslash \{w, j\}$. Let $O_S = \sum_{s \in S} O_s = \sum_{s \in S} |W|_{:,s} m_s$.
Suppose $\exists \boldsymbol{u}$ such that $\Pi_{w,j}(\boldsymbol{u} - O_S) \notin \mathbb{S}_{w,j}$.
Then at least one of the equilibrium points $\boldsymbol{x}^*_{\boldsymbol{\sigma}}$ of the regions $\boldsymbol{\sigma}_{j,O_S}$ and $\boldsymbol{\sigma}_{w,O_S}$ will automatically fall outside of it (i.e. one of the inequalities will be verified). The regions $\boldsymbol{\sigma}_{j,O_S}$ and $\boldsymbol{\sigma}_{w,O_S}$ are:

$$\boldsymbol{\sigma}_{j,O_S} = \begin{cases} \sigma_k = l & If \quad k = j \\ \sigma_k = s & If \quad k \in S \\ \sigma_k = 0 & If \quad k \neq j \text{ and } k \notin S \end{cases} \qquad \boldsymbol{\sigma}_{w,O_S} = \begin{cases} \sigma_k = l & If \quad k = w \\ \sigma_k = s & If \quad k \in S \\ \sigma_k = 0 & If \quad k \neq w \text{ and } k \notin S \end{cases} \tag{5.12}$$

Furthermore, if the vector $\boldsymbol{u} - O_S$ is <u>outside $\mathbb{S}_{w,j}$ from the $-w$ direction</u>, meaning that:

$$\begin{cases} u_w < (O_S)_w & \text{If } u_j < (O_S)_j \\ \frac{u_w - (O_S)_w}{u_j - (O_S)_j} < \min_t \left(\frac{(\boldsymbol{I}-\boldsymbol{W})_{w,t} m_t}{(\boldsymbol{I}-\boldsymbol{W})_{j,t} m_t} \right) & \text{Otherwise} \end{cases} \tag{5.13}$$

It will be true that:

1. *The equilibrium point $\boldsymbol{x}^*_{\boldsymbol{\sigma_{w,O_S}}} \notin \boldsymbol{\sigma}_{w,O_S}$.*

2. *$\forall S'$ set of inhibitory nodes, such that $w, j \notin S'$ and $S \subseteq S'$ the inequalities for a region like*

$$\boldsymbol{\sigma} = \begin{cases} \sigma_k = l & \text{If } k = w \\ \sigma_k = 0 & \text{If } k = j \\ \sigma_k = s & \text{If } k \in S' \\ \sigma_k = 0 & \text{If } k \neq w, j \text{ and } k \notin S' \end{cases} \tag{5.14}$$

   will be verified.

3. *$\forall S'$ set of inhibitory nodes such that $w \in S'$, $j \notin S'$ and $S \subseteq S'$ the inequalities for a region like:*

$$\boldsymbol{\sigma} = \begin{cases} \sigma_k = l & \text{If } k = j \\ \sigma_k = s & \text{If } k = w \\ \sigma_k = s & \text{If } k \in S' \text{ and } k \neq w \\ \sigma_k = 0 & \text{If } k \neq j \text{ and } k \notin S' \end{cases} \tag{5.15}$$

   Will be verified.

Proof. For the general result, we have that the projection $\Pi_{w,j}$ is out of the cone $\mathbb{S}_{w,j}$ meaning that:

$$\frac{u_w-(O_S)_w}{u_j-(O_S)_j} > \frac{(\boldsymbol{I}-\boldsymbol{W})_{w,t}m_t}{(\boldsymbol{I}-\boldsymbol{W})_{j,t}m_t} \;\; \forall t \quad or \quad \frac{u_w-(O_S)_w}{u_j-(O_S)_j} < \frac{(\boldsymbol{I}-\boldsymbol{W})_{w,t}m_t}{(\boldsymbol{I}-\boldsymbol{W})_{j,t}m_t} \;\; \forall t$$

$$or \;\; u_j < (O_S)_j \quad or \quad u_w < (O_S)_w \tag{5.16}$$

In particular:

$$\frac{u_w-(O_S)_w}{u_j-(O_S)_j} > \max\Big(\frac{|W|_{w,w}+1}{W_{j,w}}, \frac{W_{w,j}}{|W|_{j,j}+1}\Big) \quad or \quad \frac{u_w-(O_S)_w}{u_j-(O_S)_j} < \min\Big(\frac{|W|_{w,w}+1}{W_{j,w}}, \frac{W_{w,j}}{|W|_{j,j}+1}\Big)$$

$$or \;\; u_j < (O_S)_j \;\; or \;\; u_w < (O_S)_w \tag{5.17}$$

If that happens, then:

$$\begin{gathered}(u_w-(O_S)_w)\frac{|W|_{j,j}+1}{|W|_{w,j}} > u_j-(O_S)_j \quad or \quad (u_w-(O_S)_w) < \frac{|W|_{w,w}+1}{|W|_{j,w}}(u_j-(O_S)_j) \\ u_j < (O_S)_j \;\; or \;\; u_w < (O_S)_w\end{gathered} \tag{5.18}$$

Which it is equivalent, by inspection to one of the following sets of inequalities being held:

$$\begin{gathered}\begin{cases} u_t < \frac{|W|_{t,j}}{|W|_{j,j}+1}\big(u_j-(O_S)_j\big)+(O_S)_t+m_t & \forall t\neq j \text{ and } t\in S' \\ u_t > \frac{|W|_{t,j}}{|W|_{j,j}+1}\big(u_j-(O_S)_j\big)+(O_S)_t & \forall t\neq j \text{ and } t\notin S' \\ u_j < (O_S)_j \\ u_j > m_j(|W|_{j,j}+1)+(O_S)_j \end{cases} \\ \begin{cases} u_t < \frac{|W|_{t,w}}{|W|_{w,w}+1}\big(u_w-(O_S)_w\big)+(O_S)_t+m_t & \forall t\neq w \\ u_t > \frac{|W|_{t,w}}{|W|_{w,w}+1}\big(u_w-(O_S)_w\big)+(O_S)_t & \forall t\neq w \\ u_w < (O_S)_w \\ u_w > m_w(|W|_{w,w}+1)+(O_S)_w \end{cases}\end{gathered} \tag{5.19}$$

So the equilibrium point $\boldsymbol{x}_{\boldsymbol{\sigma}}*$ of one of the regions $\boldsymbol{\sigma}_w$ and $\boldsymbol{\sigma}_j$ will automatically fall outside.
If we now suppose that:

$$\begin{cases} u_w < (O_s)_w & \text{If } u_j < (O_s)_j \\ \frac{u_w-(O_S)_w}{u_j-(O_S)_w} < \min_t\left(\frac{(\boldsymbol{I}-\boldsymbol{W})_{w,t}m_t}{(\boldsymbol{I}-\boldsymbol{W})_{j,t}m_t}\right) & Otherwise \end{cases} \tag{5.20}$$

The proof for the other results will be:

1. The proof immediately follows the previous result, bounding the maximum value of $\frac{u_w-(O_S)_w}{u_j-(O_S)_j}$ if $u_j > (O_S)_j$ or the value of $u_w < (O_s)_w$ if $u_j < (O_s)_j$.

2. If $S' = S$ the region $\boldsymbol{\sigma}$ is equivalent to the region $\boldsymbol{\sigma}_{w,O_S}$.
   If $S \subset S'$ then the inequalities describing a valid input $\boldsymbol{u}$ for this concrete $\boldsymbol{\sigma}$ will be:

$$\begin{cases} u_t < \frac{|W|_{t,w}}{|W|_{w,w}+1}(u_w - \sum_{s\in S'}|W|_{w,s}m_s) + \left(\sum_{s\in S'}|W|_{t,s}m_s\right) + m_t & \forall t \in S' \\ u_t > \frac{|W|_{t,w}}{|W|_{w,w}+1}(u_w - \sum_{s\in S'}|W|_{w,s}m_s) + \left(\sum_{s\in S'}|W|_{t,s}m_s\right) & \forall t \notin S' \text{ and } t \neq j \\ u_j > \frac{|W|_{j,w}}{|W|_{w,w}+1}(u_w - \sum_{s\in S'}|W|_{w,s}m_s) + \left(\sum_{s\in S'}|W|_{j,s}m_s\right) & \forall t \neq j \quad (in.\ 1) \\ u_w < \sum_{s\in S'}|W|_{w,s}m_s = O_S + O_{S'\setminus S} & (in.\ 2) \\ u_w > \sum_{s\in S'}|W|_{w,s}m_s + (|W|_{w,w}+1)m_w = O_S + O_{S'\setminus S} + (|W|_{w,w}+1)m_w & (in.\ 3) \end{cases} \tag{5.21}$$

If $u_w \notin [\sum_{s\in S}|W|_{w,s}m_s, \sum_{s\in S}|W|_{w,s}m_s + (|W|_{w,w}+1)m_w] = [(O_S)_w + (O_{S'\setminus S})_w, (O_S)_w + (O_{S'\setminus S})_w + (|W|_{w,w}+1)m_w]$ then it is trivial.
However if not then considering the inequality relating u_j and u_w we have:

$$u_j - \left(\sum_{s\in S'}|W|_{j,s}m_s\right) > \frac{|W|_{j,w}}{|W|_{w,w}+1}(u_w - \sum_{s\in S'}|W|_{w,s}m_s) \tag{5.22}$$

One can observe that this inequality is equivalent to the following, with a displacement of $O_{S'\setminus S} = \sum_{s\in S'\setminus S} O_s$:

$$u_j - (O_S)_j - (O_{S'\setminus S})_j > \frac{|W|_{j,w}}{|W|_{w,w}+1}(u_w - (O_S)_w - (O_{S'\setminus S})_w) \tag{5.23}$$

So if $\Pi_{w,j}(\boldsymbol{u} - O_S) \notin \mathbb{S}_{w,j}$ in the $-w$ direction in then the inequality will be always verified. Graphically it is easy to interpret, supposing $S = \emptyset$):

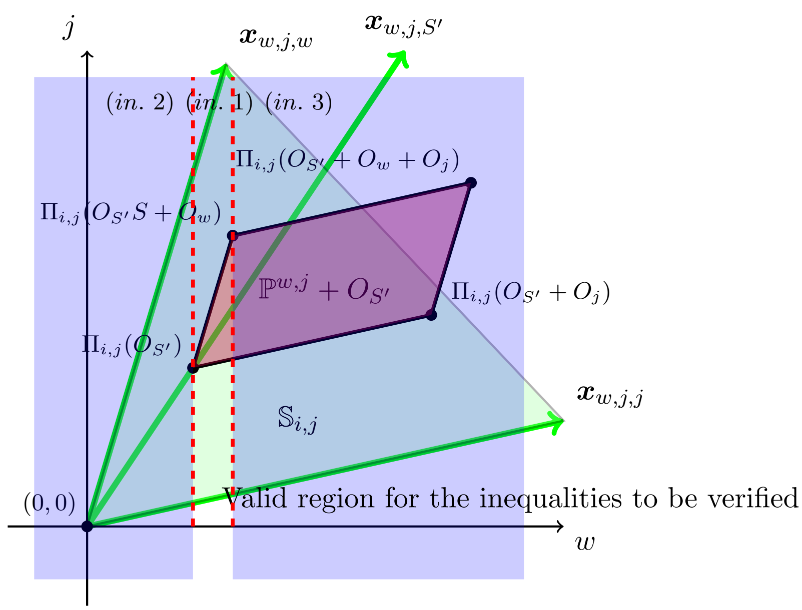


Figure 5.8: In purple it is represented the valid input values that will verify either one of the inequalities $(in.\ 1)$,$(in.\ 2)$ or $(in.\ 3)$. It is straight forward to verify that if $\Pi_{w,j}(\boldsymbol{u}) \notin \mathbb{S}_{w,j}$ in the $-w$ direction it will be automatically inside this valid region.

3. Here we are assuming that $S' \neq \emptyset$ because $w \in S'$.
Pick an arbitrary set S', including w then the inequalities that $\boldsymbol{u}$ needs to verify for the equilibrium point not to fall inside the corresponding region are:

$$\begin{cases} u_t < \frac{|W|_{t,j}}{|W|_{j,j}+1}(u_j - \sum_{s\in S'}|W|_{j,s}m_s) + \big(\sum_{s\in S'}|W|_{t,s}m_s\big) + m_t & \forall t \in S' \text{ and } t \neq w \\ u_t > \frac{|W|_{t,j}}{|W|_{j,j}+1}(u_j - \sum_{s\in S'}|W|_{j,s}m_s) + \big(\sum_{s\in S'}|W|_{t,s}m_s\big) & \forall t \notin S' \\ u_w < \frac{|W|_{w,j}}{|W|_{j,j}+1}(u_j - \sum_{s\in S'}|W|_{j,s}m_s) + \big(\sum_{s\in S'\setminus\{w\}}|W|_{w,s}m_s\big) + (|W|_{w,w}+1)m_w \\ u_j < \sum_{s\in S'}|W|_{j,s}m_s \\ u_j > \sum_{s\in S'}|W|_{j,s}m_s + (|W|_{j,j}+1)m_j \end{cases} \tag{5.24}$$

So if $u_j \notin [\sum_{s\in S'}|W|_{j,s}m_s, \sum_{s\in S'}|W|_{j,s}m_s + (|W|_{j,j}+1)m_j]$ it is trivial.
If $u_j \in [\sum_{s\in S'}|W|_{j,s}m_s, \sum_{s\in S'}|W|_{j,s}m_s + (|W|_{j,j}+1)m_j]$ then the inequality that will be always verified is:

$$u_w < \frac{|W|_{w,j}}{|W|_{j,j}+1}(u_j - \sum_{s\in S'}|W|_{j,s}m_s) + \big(\sum_{s\in S'\setminus\{w\}}|W|_{w,s}m_s\big) + (|W|_{w,w}+1)m_w \tag{5.25}$$

Let's set the minimum value for u_j, as it will be the more restrictive for the inequality. Then we have that in order to be a valid input, u_w should be superior bounded by:

$$u_w < \big(\sum_{s\in S'\setminus\{w\}}|W|_{w,s}m_s\big) + (|W|_{w,w}+1)m_w \tag{5.26}$$

However

$$\begin{aligned} \frac{u_w - (O_S)_w}{u_j - (O_S)_j} &< \min\big(\frac{|W|_{w,w}+1}{W_{j,w}}, \frac{|W|_{w,j}}{W_{j,j}+1}, \min_{t\neq w,j}(\frac{|W|_{w,t}}{W_{j,t}})\big) \\ u_w - (O_S)_w &< \min\big(\frac{|W|_{w,w}+1}{W_{j,w}}, \frac{|W|_{w,j}}{W_{j,j}+1}, \min_{t\neq w,j}(\frac{|W|_{w,t}}{W_{j,t}})\big) \sum_{s\in S'\setminus S}|W|_{j,s}m_s \end{aligned} \tag{5.27}$$

Considering each term individually, for each $s \in S'\setminus S$:

$$\min\big(\frac{|W|_{w,w}+1}{W_{j,w}}, \frac{|W|_{w,j}}{W_{j,j}+1}, \min_{t\neq w,j}\frac{|W|_{w,t}}{W_{j,t}}\big)|W|_{j,s}m_s \leq |W|_{w,s}m_s \tag{5.28}$$

So trivially:

$$u_w < \big(\sum_{s\in S'\setminus\{w\}}|W|_{w,s}m_s\big) + (|W|_{w,w}+1)m_w \tag{5.29}$$

□

Definition 5.1.6. ***(Cone chain).***
Let (u_w/u_j) symbolize that $\Pi_{w,j}(\boldsymbol{u} - O_S)$ is out of the cone $\mathbb{S}_{w,j}$ through $-w$ direction $\forall S$ set of inhibitory nodes in saturation different that w, j.
A cone chain will be a set nodes $t_1, \ldots, t_m$ verifying the following:

$$\begin{cases} (u_{t_2}/u_{t_1}), \quad (u_{t_3}/u_{t_1}), \quad \ldots \quad , \quad (u_{t_m}/u_{t_1}) \\ (u_{t_3}/u_{t_2}), \quad \ldots \quad , \quad (u_{t_m}/u_{t_2}) \\ \quad \vdots \\ (u_{t_m}/u_{t_{m-1}}) \end{cases} \tag{5.30}$$

Lemma 5.1.7. ***(Chain lemma).***
Let $\boldsymbol{u}$ such that it admits a cone chain of inhibitory nodes $\{i_1, \ldots, i_m\}$. If also $\exists u_k > \frac{|W|_{k,i_1}}{|W|_{i_1,i_1}+1} u_{i_1}$, for some k belonging to the excitatory nodes, and for i_m then $u_{i_m} < \sum\limits_{t \in Inhibitory \setminus i_m} |W|_{i_m,t} m_t$ then all the equilibrium candidates of the regions where no excitatory node is in positive saturation state will fall outside them.

Proof. The inequalities for the region $\boldsymbol{\sigma}_{i_1} = (0, \ldots, 0, \underbrace{l}_{i_1}, 0, \ldots, 0)$ will be verified as:

$$u_k > \frac{|W|_{k,i_1}}{|W|_{i_1,i_1} + 1} u_{i_1} \tag{5.31}$$

Now let $(L_{i_r}, \mathbf{0})$, (L_{i_r}, S_{i_t}) and $(L_{i_t}, 0_{i_r})$ respectively be:

$$(L_{i_r}, \mathbf{0}) = \begin{cases} \sigma_k = l & \text{If } k = i_r \\ \sigma_k = 0 & \text{Otherwise} \end{cases} \qquad (L_{i_r}, S_{i_t}) = \begin{cases} \sigma_k = l & \text{If } k = i_r \\ \sigma_k = s & \text{If } k = i_t \\ \sigma_k = \{0, s\} & \text{Otherwise} \end{cases}$$
$$(L_{i_t}, 0_{i_r}) = \begin{cases} \sigma_k = l & \text{If } k = i_t \\ \sigma_k = 0 & \text{If } k = i_r \\ \sigma_k = \{0, s\} & \text{Otherwise} \end{cases} \tag{5.32}$$

Now, using the chain condition on i_1, and the lemma 5.2.3 we can affirm that the following regions will not contain its equilibrium points:

$$\begin{array}{llll} (L_{i_2}, \mathbf{0}) & (L_{i_3}, \mathbf{0}) & \ldots & (L_{i_m}, \mathbf{0}) \\ (L_{i_1}, S_{i_2}) & (L_{i_1}, S_{i_3}) & \ldots & (L_{i_3}, 0_{i_1}) \\ (L_{i_2}, 0_{i_1}) & (L_{i_1}, S_{i_m}) & \ldots & (L_{i_m}, 0_{i_1}) \end{array}$$

Using the chain condition on i_2 and lemma 5.2.3 again we can affirm that the following regions will not contain its equilibrium points:

$$\begin{array}{lll} (L_{i_3}, \mathbf{0}) & \dots & (L_{i_m}, \mathbf{0}) \\ (L_{i_2}, S_{i_3}) & \dots & (L_{i_2}, S_{i_m}) \\ (L_{i_3}, 0_{i_2}) & \dots & (L_{i_m}, 0_{i_2}) \end{array}$$

Repeating this process until the node i_{m-1}. Finally for the node i_{m-1} we will have:

$$\begin{array}{l} (L_{i_m}, \mathbf{0}) \\ (L_{i_{m-1}}, S_{i_m}) \\ (L_{i_m}, 0_{i_{m-1}}) \end{array}$$

By inspection, the only region for which the equilibrium point is not guaranteed to fall outside the corresponding region after applying the chain condition to all the permutation of the nodes is $\boldsymbol{\sigma} = (L_{i_m}, S_{i_1}, S_{i_2}, \dots, S_{m-1})$. However as:

$$u_{i_m} < \sum_{t \in Inhibitory \backslash i_m} |W|_{i_m,t} m_t \tag{5.33}$$

The set of inequalities defining a valid input region will be verified. □

Lemma 5.1.8. ***(Sufficient condition for u** $\notin \mathbb{P}^L$, $|L| \geq 2$**).***
Let L be a set of inhibitory nodes such that $|L| \geq 2$.
Let k be an arbitrary inhibitory node.
If $u \in \mathbb{R}^N$ such that $u_i < 0$ for all $i \in L \backslash \{k\}$ then the equilibrium point will not fall in its corresponding switching region for all the switching regions of the form $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ s.t $\mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{l,i} \mid \geq 2$ and $\mid \Sigma^{0,e} \mid = n$.

Proof. Using the corollary (5.1.3) for $|L| > 1$ the inequalities will be always verified, as for each pair and each projection, one term will be always negative implying the projection to be outside of the cone. □

Lemma 5.1.9. ***(Sufficient condition for regions such that*** $\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ *s.t* $\mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ ***and*** $\mid \Sigma^{s,i} \mid + \mid \Sigma^{l,i} \mid \geq 1$ ***).***
If $u \in \mathbb{R}^N$ such that $u_j < 0$ $\forall j \in L$ then the equilibrium point will not fall in its corresponding switching region for all the regions of the form $\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ s.t $\mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{l,i} \mid \geq 0$.

Proof. For the regions where $|L| > 1$ it is a direct consequence of lemma (5.1.8).
Regarding the case where $|L| = 1$, if no other nodes are in positive saturation, then the inequalities will be trivially verified as the set of inequalities look like, for every j:

$$\begin{cases} u_1 > \frac{|W|_{1,j}}{|W|_{j,j}+1} u_j \\ u_k > \frac{|W|_{k,j}}{|W|_{j,j}+1} u_j \\ u_j < 0 & (in.\ negative) \\ u_j > m_j(|W|_{j,j} + 1) \end{cases} \tag{5.34}$$

And the inequalities of the form of (*in. negative*) will always hold.
If the set S of inhibitory saturated nodes is not empty, then the form of (*in. negative*) will be:

$$u_j < \sum_{s\in S} |W|_{j,s} m_s \tag{5.35}$$

So it will be also verified.
For $|L| = 0$ it means that $|\Sigma^{s,i}| > 0$. Then for the j node in positive saturation we will have that:

$$u_j < \sum_{s\in S} |W|_{j,s} m_s + m_j \tag{5.36}$$

□

Lemma 5.1.10. ***(Sufficient condition for regions such that*** $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ *s.t* $|\Sigma^{s,i}| + |\Sigma^{0,i}| + |\Sigma^{l,i}| = m$ ***and*** $|\Sigma^{l,i}| \geq 1$ ***and*** $|\Sigma^{0,e}| = n$ ***).***
If for every $\Pi_{i,j}(\boldsymbol{u}) \notin \mathbb{S}_{i,j}$ *and* $\exists u_k$*, k excitatory node, such that* $u_k > \frac{|W|_{k,j}}{|W|_{j,j}+1} u_j$ *and* $\forall i \in \{n, \ldots, m+n\}$ $u_i < \min\limits_{t\in Inhibitory} (W_{i,t} m_t)$ *then the equilibrium point will not fall inside its corresponding switching zone for all the regions of the form* $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ *s.t* $|\Sigma^{s,i}| + |\Sigma^{0,i}| + |\Sigma^{l,i}| = m$ *and* $|\Sigma^{l,i}| \geq 0$.

Proof. For the regions where $|L| > 1$ it is a direct consequence of lemma (5.1.3).
For the case where $|L| = 1$ it is a direct consequence of the (5.1.7).

□

5.2 Results for regions with arbitrary excitatory nodes in positive saturation

Lemma 5.2.1. ***(Parallelogram theorem with a displaced origin).***
Let E be the set of excitatory nodes and let $K \subseteq E$.
Let L be a fixed set of inhibitory nodes in linear state, such that $|L| \geq 2$.
Let $Q_{i,j}$, $t_{i,j}$, $T_{i,j}$, $O_{t_{i,j}}$ *be described as in theorem (5.1.1).*
Let O_k, for any $k \in E$, be described as:

$$O_k = \begin{cases} (O_k)_t = 0 & \text{If } t \notin L \\ (O_k)_t = -|W|_{t,k} m_k & \text{If } k \in L \end{cases} \tag{5.37}$$

Let $\boldsymbol{v} = (\boldsymbol{\Sigma}^l \boldsymbol{u} + \boldsymbol{\Sigma}^{s,K} \boldsymbol{m})$. *If* $\exists\ \Pi_{i,j}(\boldsymbol{u} - \sum_{k\in K\subseteq E} O_k - \sum\limits_{\forall t_{i,j}} \Gamma_{t_{i,j}} O_{t_{i,j}}) \notin \mathbb{Q}_{i,j}\ \forall \Gamma_{t_{i,j}} \in [0,1]$, *then* $\boldsymbol{v} \notin \mathbb{P}^l + \sum_{k\in K\subseteq E} O_k$, *so some of the inequalities of the form (in.* 1) *or (in.* 2) *for* $\boldsymbol{\sigma} = (\Sigma^{s,K}, \Sigma^{0,E-K}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ *s.t* $|\Sigma^{s,i}| + |\Sigma^{0,i}| + |\Sigma^{l,i}| = m$ *and* $|\Sigma^{l,i}| > 1$ *and such that* $|\Sigma^{s,K}| + |\Sigma^{0,E-K}| = n$ *will be held.*

Proof. The proof is equivalent to (5.1.1) but with a change on the origin. The change of the origin is due to the excitatory nodes being now in positive saturated state, so we will only proof the

equivalence
In order to proof it let $\mathbb{U}' = U_1' \times U_2' \times \cdots \times U_{m+n}'$ be, where U_i is:

$$U_i' = \begin{cases} \{m_k\} & \text{If } i \in K \\ \{0\} & \text{If } i \in E \backslash K \\ [0, m_i] & \text{If } (\Sigma^l)_{i,i} = 1 \\ \{m_i\} & \text{If } (\Sigma^l)_{i,i} = 0 \text{ and } (\Sigma^{s,i})_{i,i} = 1 \\ \{0\} & \text{Otherwise} \end{cases}$$

and let $\mathbb{U} = U_1 \times U_2 \times \cdots \times U_{m+n}$ be, where U_i is:

$$U_i = \begin{cases} \{0\} & \text{If } i \in E \\ [0, m_i] & \text{If } (\Sigma^l)_{i,i} = 1 \\ \{m_i\} & \text{If } (\Sigma^l)_{i,i} = 0 \text{ and } (\Sigma^{s,i})_{i,i} = 1 \\ \{0\} & \text{Otherwise} \end{cases}$$

So we have that:

$$\begin{aligned} \mathbb{P}^{L'} &= \boldsymbol{K}^{-1}(\mathbb{U}') = (\boldsymbol{I} - \Sigma^L \boldsymbol{W})(\Sigma^L \boldsymbol{u} + \Sigma^{s,i}\boldsymbol{m} + \Sigma^{s,e}\boldsymbol{m}) = \\ &= \sum_{k\in K} m_k (\boldsymbol{I} - \Sigma^L \boldsymbol{W})_{:,k} + (\boldsymbol{I} - \Sigma^L \boldsymbol{W})(\Sigma^L \boldsymbol{u} + \Sigma^{e,s}\boldsymbol{m}) = \sum_{k\in K} O_k + \boldsymbol{K}^{-1}(\mathbb{U}) = \sum_{k\in K} O_k + \mathbb{P}^L \end{aligned} \tag{5.38}$$

So we can affirm that $\boldsymbol{v} \notin \mathbb{P}^L + \sum_{k\in K\subseteq E} O_k \Leftrightarrow \boldsymbol{v} - \sum_{k\in K\subseteq E} O_k \notin \mathbb{P}^L$ which is equivalent that $\exists$ $\Pi_{i,j}(\boldsymbol{u} - \sum_{k\in K\subseteq E} O_k - \sum\limits_{\forall t_{i,j}} \Gamma_{t_{i,j}} O_{t_{i,j}}) \notin \mathbb{Q}_{i,j}$ $\forall \Gamma_{t_{i,j}} \in [0,1]$. □

Corollary 5.2.2. ***(Cone approach with displaced origin).***
Let L be a set of inhibitory nodes such that $|L| \geq 2$.
Let $T_{i,j} \subseteq L$ such that $|T| \geq 2$.
Let $\mathbb{S}_{i,j}$, with $i,j \in T_{i,j} \subseteq L$ be the convex cone defined by the 2-dimensional vectors as before.
Let $\Pi_{i,j}$ be the projection to the i,j plane.
Let E be the set of excitatory nodes and let $K \subseteq E$.
Let $O_K = \sum_{k\in K} O_k$.
If there exists a pair i,j such that $\Pi_{i,j}(\boldsymbol{u} - O_K) \notin \mathbb{S}_{i,j}$ then $(\Sigma^T \boldsymbol{u}) \notin \mathbb{P}^T + O_K$ for every $T \subseteq L$ such that $i,j \in T$ and $|T| \geq 2$.
Furthermore, let S be the set of positive saturated nodes such that $S \subseteq L\backslash T$. Let $\mathbb{P}^T + O_K + \sum_{s\in S} O_s$ be the polyhedron $\mathbb{P}^T + O_K$ translated by $\sum_{s\in S} O_s$. Then, it is also true that $(\Sigma^T \boldsymbol{u} + \Sigma^S \boldsymbol{m}) \notin \mathbb{P}^T + O_K + \sum_{s\in S} O_s$

Lemma 5.2.3. ***(Consequences of the cone approach for $|L| = 1$ with a displaced origin).*** *Let w, j be two inhibitory nodes.*
Let E be the set of excitatory nodes and let $K \subseteq E$ and let $O_K = \sum_{k \in K} O_k$.
Let S be a set of inhibitory nodes such that $w, j \notin S$ and let $O_S = \sum_{k \in K} O_k$.
Suppose $\exists \boldsymbol{u}$ and a pair of nodes $\{w, j\}$ such that $\Pi_{w,j}(\boldsymbol{u} - O_K - O_S) \notin \mathbb{S}_{w,j}$.
The equilibrium points $\boldsymbol{x}^*_{\boldsymbol{\sigma}}$ of the regions $\boldsymbol{\sigma}_{j,K,S}$ and $\boldsymbol{\sigma}_{w,K,S}$ will automatically fall outside of it (i.e. one of the inequalities will be verified). The regions $\boldsymbol{\sigma}_{j,K,S}$ and $\boldsymbol{\sigma}_{w,K,S}$ are:

$$\boldsymbol{\sigma}_j = \begin{cases} \sigma_t = s & If \quad t \in K, S \\ \sigma_t = l & If \quad t = j \\ \sigma_t = 0 & If \quad t \neq j \text{ and } t \notin K \end{cases} \qquad \boldsymbol{\sigma}_w = \begin{cases} \sigma_t = s & If \quad t \in K, S \\ \sigma_t = l & If \quad t = w \\ \sigma_t = 0 & If \quad t \neq w \text{ and } t \notin K \end{cases} \tag{5.39}$$

Furthermore, if the vector is outside of the cone from the $-w$ direction, meaning that:

$$\frac{u_w - (O_K)_w - (O_S)_w}{u_j - (O_K)_j - (O_S)_j} < \min_t \left(\frac{(\boldsymbol{I} - \boldsymbol{W})_{w,t} m_t}{(\boldsymbol{I} - \boldsymbol{W})_{j,t} m_t} \right) \tag{5.40}$$

Then it will be true that:

1. *The equilibrium point $\boldsymbol{x}^*_{\boldsymbol{\sigma_{w,K,S}}} \notin \boldsymbol{\Omega}_{\boldsymbol{\sigma}_{w,K,S}}$*
2. *$\forall S'$ set of inhibitory nodes, such that $S \subseteq S'$, such that $w, j \notin S'$ the inequalities for a region like*

$$\boldsymbol{\sigma} = \begin{cases} \sigma_t = s & \text{If } t \in K, S \\ \sigma_t = l & \text{If } t = w \\ \sigma_t = 0 & \text{If } t = j \\ \sigma_t = s & \text{If } t \in S' \backslash S \\ \sigma_t = 0 & \text{If } t \neq w, j \text{ and } t \notin S' \text{ and } t \notin K \end{cases} \tag{5.41}$$

   will be verified.

3. *$\forall S'$ set of inhibitory nodes such that $w \in S$ and $S \subset S'$ and $j \notin S$ the inequalities for a region like:*

$$\boldsymbol{\sigma} = \begin{cases} \sigma_t = s & \text{If } t \in K, S \\ \sigma_t = l & \text{If } t = j \\ \sigma_t = s & \text{If } t = w \\ \sigma_t = s & \text{If } t \in S' \backslash S \text{ and } t \neq w \\ \sigma_t = 0 & \text{If } t \neq j \text{ and } t \notin S' \text{ and } t \notin K \end{cases} \tag{5.42}$$

   Will be verified.

Proof. The proof is equivalent to the proof for the lemma (5.2.3). □

Definition 5.2.4. ***(Chain with displaced origin).***
Let $(u_w/u_j)_K$ symbolize that $\Pi_{w,j}(\boldsymbol{u} - O_S - O_K)$ is out of the cone $\mathbb{S}_{w,j}$ through $-w$ direction $\forall S$ set of inhibitory nodes in saturation different that w, j and for the displaced origin K.
Then a cone chain with displaced origin K will be a set nodes $t_1, \ldots, t_m$ verifying the following

$$\begin{cases} (u_{t_2}/u_{t_1})_K, \quad (u_{t_3}/u_{t_1})_K \quad ,\ldots \quad , \quad (u_{t_m}/u_{t_1})_K \\ (u_{t_3}/u_{t_2})_K, \quad ,\ldots \quad , \quad (u_{t_m}/u_{t_2})_K \\ \quad \vdots \\ (u_{t_m}/u_{t_{m-1}})_K \end{cases} \tag{5.43}$$

Lemma 5.2.5. ***(Chain lemma with displaced origins).***
Let E be the set of excitatory nodes and let $K \subseteq E$.
Let $O_K = \sum_{k \in K} O_k$, be the displaced origin.
Choosing a chain with displaced origin O_K of inhibitory nodes $\{i_1, \ldots, i_m\}$ such that the inequalities of the region $(l_j, O_K, \mathbf{0})$ are verified and $u_{i_m} < (O_K)_{i_m} + \sum\limits_{t \in Inhibitory \backslash i_m} |W|_{i_m,t} m_t$ then all the equilibrium points of the regions where the nodes excitatory nodes K are in positive saturation state and that there is at least one node in linear state, will fall outside its corresponding zone.

Proof. The proof is equivalent to the proof for the lemma (5.1.7) but considering the new origin O_K.

□

Corollary 5.2.6. ***(Cone conclusions for arbitrary networks).***
Let E be the set of excitatory nodes.
Let $K \subseteq E$ be a subset of excitatory nodes.
Let $O_K = \sum_{k \in K} O_k$, be the displaced origin defined by K.
If $\forall K$ the hypothesis for the cone approach with displaced origins are held and there $\exists$ a single chain verifying all the conditions for the chain lemma with displaced origin for every O_K, even for non-displaced origin, then all the equilibrium points of the regions where there is at least one inhibitory node in linear state will fall outside its corresponding zone.

Lemma 5.2.7. ***(Sufficient condition for $\boldsymbol{u} \notin \mathbb{P}^L + O_K$, $|L| \geq 2$).***
Let L be a set of inhibitory nodes such that $|L| \geq 2$.
Let $K \subseteq E$ be a set of excitatory nodes in positive saturation.
Let j be an arbitrary inhibitory node.
If $u \in \mathbb{R}^N$ such that $u_i < (O_K)_i$ for all $i \in L \backslash \{j\}$ then the equilibrium point will not fall in its corresponding switching region for all the switching regions of the form $\boldsymbol{\sigma} = (\Sigma^{0,E-K}, \Sigma^{s,K}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ s.t $\mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{l,i} \mid \geq 2$ and $\mid \Sigma^{0,E-K} \mid + \mid \Sigma^{s,K} \mid = n$.

Chapter 6

Networks nE - mI

This section will discuss the non-equilibria conditions for the networks following the dynamics (2.7),(2.8) with an arbitrary number (n) of excitatory nodes and an arbitrary number (m) of inhibitory nodes.
It will first be discussed two different sets of conditions to impose on $\boldsymbol{W}$ in order to verify assumption (2), regarding the instability of the switching regions containing an excitatory node in linear state. Then, three different sets of sufficient conditions on the structure of $\boldsymbol{W}$ will be provided for arbitrary nE-mI networks and for them not to have any stable equilibria.
Furthermore, some partial results for the particular case of networks 1E-mI will also be exposed.

6.1 Conditions for inestability of excitatory nodes

Theorem 6.1.1. ***(Conditions on the network structure*** $(\boldsymbol{W})$ ***to provide instability for excitatory nodes).***
Let $\boldsymbol{W}$ *be the matrix of the average synaptic connections describing the dynamics of the system. As exposed before, the form of* $\boldsymbol{W}$ *will be:*

$$\boldsymbol{W} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} & -b_{1,1} & \dots & -b_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & -b_{n,1} & \dots & -b_{n,m} \\ c_{1,1} & \dots & c_{1,n} & -d_{1,1} & \dots & -d_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m,1} & \dots & c_{m,n} & -d_{m,1} & \dots & -d_{m,m} \end{bmatrix} \tag{6.1}$$

With a*,* b*,* c $d \in \mathbb{R}_{\geq 0}$*.*
Then, it is sufficient for assumption (2) to be verified (i.e. that all the switching regions where at

least one excitatory node is in linear state) that either:

$$\begin{cases} \forall i \in \{1, \ldots, n\} \\ \quad a_{i,i} > m + 1 + \sum_{j \in \{n+1,\ldots,n+m\}} d_{j,j} \end{cases} \tag{6.2}$$

Or that:

$$\begin{cases} \forall i \in \{1, \ldots, n\} \textit{ and } \forall k \in \{n+1, \ldots, m+n\} \\ \qquad a_{i,i} > (d_{k,k} + 2) \\ \forall E \subseteq \{1, \ldots, n\} \textit{ and } \forall I \subseteq \{n+1, \ldots m+n\} \textit{ such that } |I| > 1 \textit{ and } |E| \geq 1 \\ \qquad \sum_{\substack{e,i \in E \\ e \neq i}} \big((1-a_{e,e})(1-a_{i,i}) - a_{i,e}a_{e,i}\big) + \sum_{\substack{k \in I \\ i \in E}} \big((1-a_{i,i})(1+d_{k,k}) + b_{i,k}c_{k,i}\big) \\ \qquad + \sum_{\substack{k,j \in I \\ j \neq k}} \big((d_{k,k}+1)(d_{j,j}+1) - d_{k,j}d_{j,k}\big) < 0 \end{cases} \tag{6.3}$$

Proof. For both sets of conditions we will consider only those switching regions where there is at least one excitatory node in linear mode and proof that the corresponding equilibrium candidate is unstable.

- For the first set, consider a region $\boldsymbol{\sigma}$ with r excitatory nodes in linear state. Let $\boldsymbol{\Pi}'_{\boldsymbol{\sigma}}$ the permutation matrix such that $\boldsymbol{\Pi}'\boldsymbol{\sigma} = [\mathbf{0}_{n-r}, \underbrace{l, \ldots, l}_{\text{r}}, \sigma_{n+1}, \ldots, \sigma_{n+m}]^T$, meaning that all the linear excitatory nodes and all the inhibitory ones are matched with the right bottom of the system matrix.
  Then, the coefficients of this permutation of the matrix $-\boldsymbol{I} + \boldsymbol{\Sigma}^l \boldsymbol{W}$ in the region $\Omega_{\boldsymbol{\sigma}}$ will satisfy:

  $$\boldsymbol{\Pi}'_{\boldsymbol{\sigma}}(-\boldsymbol{I} + \boldsymbol{\Sigma}^l \boldsymbol{W})\boldsymbol{\Pi}'^T_{\boldsymbol{\sigma}} = \begin{bmatrix} -\boldsymbol{I}_{n-r} & 0 \\ * & \boldsymbol{P} \end{bmatrix}$$

  where:

  $$\boldsymbol{P} = \begin{bmatrix} -\boldsymbol{I}_r + \boldsymbol{a}_r & * \\ * & -\boldsymbol{I}_m - \boldsymbol{\Sigma}^l_{2,2}\boldsymbol{d} \end{bmatrix}$$

  With that, the eigenvalues of $-\boldsymbol{I} + \boldsymbol{\Sigma}^l \boldsymbol{W}$ consists of (-1) with multiplicity $(n - r - l)$ and the eigenvalues of $\boldsymbol{P}$. Therefore, as $r > 0$, then the equilibrium point of $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ is unstable since:

  $$\begin{aligned} tr(\boldsymbol{P}) &= \sum_{i \in L}(a_{ii} - 1) + \sum_{j \in T}(-1 - \Sigma_{j+n,j+n} d_{j+n,j+n}) \geq \sum_{i \in L}(a_{ii} - 1) + \sum_{j=n+1}^{n+m}(-1 - d_{jj}) \\ &\geq \sum_{i \in L}(a_{ii} - 1) - m - \sum_{j=n+1}^{n+m} d_{jj} \geq \min(a_{ii}) - 1 - m - \sum_{j=n+1}^{n+m} d_{jj} \overset{(6.2)}{>} 0 \end{aligned}$$

- For the second set of conditions, we will also consider an arbitrary $\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ containing at least one excitatory node in linear state.

If $\exists!\ r$ excitatory node in linear them the eigenvalues of $-\boldsymbol{I}+\boldsymbol{\Sigma}^l\boldsymbol{W}$ will be (-1) with multiplicity $(N-1)$ and the eigenvalues of $\boldsymbol{P}$. As, for an arbitrary $k \in \{n+1, \dots, n+m\}$, $a_{i,i} > d_{k,k}+2 > 2$ then $((a_{i,i}) - 1) > 0$, so $Tr(\boldsymbol{P}) = a_{r,r} - 1 > 0$ and the equilibrium point will be unstable.
If $\boldsymbol{\Omega_\sigma}$ contains only two linear nodes, one excitatory node, i, and one inhibitory node, k, then the eigenvalues of $-\boldsymbol{I} - \boldsymbol{\Sigma}^l\boldsymbol{W}$ will be (-1) with multiplicity $(N-2)$ and the eigenvalues of the following the submatrix $\boldsymbol{P}$:

$$\boldsymbol{P} = \begin{bmatrix} a_{i,i} - 1 & -b_{i,k} \\ c_{k,i} & -1 - d_{k,k} \end{bmatrix} \tag{6.4}$$

As $Tr(\boldsymbol{P}) = a_{i,i} - d_{k,k} - 2$, and our assumption is that $a_{i,i} > d_{k,k} + 2$ then $Tr(\boldsymbol{P}) > 0$. This implies that at least one eigenvalue will have a positive real part.
Let's consider now a switching region $\boldsymbol{\Omega_\sigma}$ with an arbitrary set of excitatory nodes E such that $|E| \geq 1$ and an arbitrary set of inhibitory nodes I such that $|I| > 1$.
Consider again a permutation matrix $\boldsymbol{\Pi}$ that matches the nodes in linear to the right bottom of the matrix, so $\boldsymbol{\Pi_\sigma \sigma} = [\boldsymbol{0}_{n-|E|}, \boldsymbol{0}_{m-|I|}, \underbrace{l, \dots, l}_{E}, \underbrace{l, \dots, l}_{I}]$.

Then the system will be:

$$\boldsymbol{\Pi_\sigma}(-\boldsymbol{I} + \boldsymbol{\Sigma}^l\boldsymbol{W})\boldsymbol{\Pi}_{\boldsymbol{\sigma}}^T = \begin{bmatrix} -\boldsymbol{I}_{n+m-|E|-|I|} & 0 \\ * & \boldsymbol{P}^l \end{bmatrix} \tag{6.5}$$

So the eigenvalues of the system will be (-1) of multiplicity $(n + m - |E| - |I|)$ and the eigenvalues of $\boldsymbol{P}^l$.
Let's consider the Characteristic Polynomial of $\boldsymbol{P}^l$:

$$Char(\boldsymbol{P}^l - \lambda\boldsymbol{I}) = (-1)^{|N|}\lambda^{|N|} + (-1)^{|N|-1}K_{|N|-1}\lambda^{|N|-1} + \cdots + (-1)K_1\lambda + det(\boldsymbol{P}^l) \tag{6.6}$$

Where $K_k = \sum_{\forall j} C_{k,j}$ where $C_{k,j}$ are the determinants of all the principal minors of size $|N| - k$.
Taking into account this, one can see that:

$$\begin{aligned} K_{|N|-2} = &\sum_{\substack{e,i \in E \\ e \neq i}} \big((1 - a_{e,e})(1 - a_{i,i}) - a_{i,e}a_{e,i}\big) + \sum_{\substack{k \in I \\ i \in E}} \big((1 - a_{i,i})(1 + d_{k,k}) + b_{i,k}c_{k,i}\big) \\ &+ \sum_{\substack{k,j \in I \\ j \neq k}} (d_{k,k} + 1)(d_{j,j} + 1) - d_{k,j}d_{j,k} < 0 \end{aligned} \tag{6.7}$$

Then it will always hold that $sgn((-1)^{|N|}) \neq sgn((-1)^{|N|-1}K_{|N|-2})$. Using Routh criteria this implies that it always $\exists\lambda_i$ for which $Re(\lambda_i) > 0$, meaning that the equilibrium point of the corresponding switching region will be unstable.

□

6.2 Results for arbitrary networks nE - mI

Corollary 6.2.1. ***(Sufficient conditions for the existence of oscillations in a n-excitatory - m-inhibiroty network.).***
Consider the dynamics (2.7)-(2.8), with n excitatory nodes and m inhibitory nodes, and assume $\boldsymbol{W}$ verifies assumption (2). For the system to not have any stable equilibria [1]*, it is it is sufficient for the system that both $\boldsymbol{m}$ and $\boldsymbol{W}$ verify the following conditions:*

$$\begin{cases} \exists k \in \{1,\dots,n\} \textit{ and } \exists j \in \{n+1,\dots,n+m\} \textit{ such that} & \\ \quad (|W|_{k,k}-1)(1+|W|_{j,j}) < |W|_{j,k}|W|_{k,j} & \textit{(Sufficient condition 1)} \\ \quad (|W|_{k,k}-1)m_k < |W|_{k,j}m_j \quad \textit{or} \quad (|W|_{j,j}+1)m_j > |W|_{j,k}m_k & \textit{(Sufficient condition 2A/2B)} \\ \quad \forall i \in \{n+1,\dots,n+m\} \textit{ and } i \neq j & \textit{(Sufficient condition 3)} \\ \qquad (|W|_{k,i}(|W|_{j,j}+1) - |W|_{k,j}|W|_{j,i} > 0) \quad \textit{or} & \\ \qquad (|W|_{j,i}m_i > (|W|_{j,j}+1)m_j) & \end{cases} \tag{6.8}$$

Proof. To prove the existence of oscillations under these sufficient conditions a constructive proof will be presented by finding a $\boldsymbol{u}$ for which no region will contain its equilibrium candidate.
Let j and k be an inhibitory and excitatory node for which the sufficient conditions are held. Then two sets of different conditions for $\boldsymbol{u}$ are going to be exposed.
<u>Set A</u>: Let $\boldsymbol{u}$ be such that:

$$\begin{cases} for\ j & \\ \quad \left(\frac{(|W|_{k,k}-1)(|W|_{j,j}+1)}{|W|_{k,j}} - |W|_{j,k}\right)m_k < u_j < \min\left(0, -|W|_{j,k}m_k + (1+|W|_{j,j})m_j\right) & (Cond.\ 1A) \\ for\ e \neq k \text{ and } e \in \{1,\dots,n\} & \\ \quad u_e < \min\Big(m_e + \sum_{g\in\{1,\dots,n\}} \big(-|W|_{e,g}m_g\big) + |W|_{e,j}m_j\ , & \\ \qquad \frac{(|W|_{e,j}u_j - \left[(|\boldsymbol{W}|_{e,1:n} - \boldsymbol{I}_n)(|W|_{j,j}+1) - |W|_{e,j}|\boldsymbol{W}|_{j,1:n}\right]^+ \boldsymbol{m}_e)}{(|W|_{j,j}+1)}\Big) & (Cond.\ 2A) \\ for\ i \neq j \text{ and } i \in \{n+1,\dots,m+n\} & \\ \quad u_i < -\sum_{e\in\{1,\dots,n\}} |W|_{i,e}m_e & (Cond.\ 3A) \\ for\ k & \\ \quad 0 < u_k < \left(1 - |W|_{k,k} + \frac{|W|_{k,j}|W|_{j,k}}{1+|W|_{j,j}}\right)m_k + \frac{|W|_{k,j}}{(1+|W|_{j,j})}u_j & (Cond.\ 4A) \end{cases} \tag{6.9}$$

[1] The system will have unstable equilibrium for the switching regions where the excitatory node is in linear state and it will not have equilibrium, as the equilibrium point will fall outside of the switching region, for the other regions

Set B: Let $\boldsymbol{u}$ be such that:

$$\begin{cases} for\ j \\ \quad 0 < u_j < \min\Big(-|W|_{j,k}m_k + (1+|W|_{j,j})m_j , \min_{\substack{i\in\{n+1,\dots,m+n\}\\ i\neq j}} (|W|_{j,i}m_i)\Big) & (Cond.\ 1B) \\ for\ e \neq k \text{ and } e \in \{1,\dots,n\} \\ \quad u_e < \min\Bigg(m_e + \sum_{g\in\{1,\dots,n\}} \big(-|W|_{e,g}m_g\big) + |W|_{e,j}m_j\ , \\ \qquad \frac{(|W|_{e,j}u_j - \big[(|\boldsymbol{W}|_{e,1:n} - \boldsymbol{I}_n)(|W|_{j,j}+1) - |W|_{e,j}|\boldsymbol{W}|_{j,1:n}\big]^+ \boldsymbol{m}_e)}{(|W|_{j,j}+1)}\Bigg) & (Cond.\ 2B) \\ for\ i \neq j \text{ and } i \in \{n+1,\dots,m+n\} \\ \quad u_i < -\sum_{e\in\{1,\dots,n\}} |W|_{i,e}m_e & (Cond.\ 3B) \\ for\ k \\ \quad \frac{|W|_{k,j}}{(1+|W|_{j,j})}u_j < u_k < (1 - |W|_{k,k} + \frac{|W|_{k,j}|W|_{j,k}}{1+|W|_{j,j}})m_k + \frac{|W|_{k,j}}{(1+|W|_{j,j})}u_j & (Cond.\ 4B) \end{cases} \tag{6.10}$$

For the Set A, the sufficient conditions *(Sufficient condition 1)* and *(Sufficient condition 2A)*, using $(|W|_{k,k}-1)m_k < |W|_{k,j}m_j$ are used to ensure room for u_k and for u_j. The room for $(Cond.\ 1A)$ will come from:

$$\begin{aligned} \Big(\frac{(|W|_{k,k}-1)(|W|_{j,j}+1)}{|W|_{k,j}} - |W|_{j,k}\Big)m_k &< -|W|_{j,k}m_k + (1+|W|_{j,j})m_j \Big) \\ \frac{(|W|_{k,k}-1)(|W|_{j,j}+1)}{|W|_{k,j}}m_k &< (1+|W|_{j,j})m_j \\ (|W|_{k,k}-1)m_k &< |W|_{k,j}m_j \end{aligned}$$

For the Set B, the sufficient conditions *(Sufficient condition 1)* and *(Sufficient condition 2B)*, $(|W|_{j,j}+1)m_j > |W|_{j,k}m_k$, are also used to ensure room for u_k and for u_j.

In heuristic way, for every $\boldsymbol{\sigma}$ it will be check that the equilibrium does not fall into its corresponding region if the previous mentioned conditions hold. For each set, the specific way of verifying the equations will be

- $\boldsymbol{\sigma} = (0, 0, \dots, 0)$
  The union set of inequalities that must be verified for this $\boldsymbol{\sigma}$ are the following ones found in (4.7), which will be immediately verified by $u_k > 0$ because of $(cond.\ 4A)$ and $(cond.\ 4B)$.

- $\boldsymbol{\sigma} = (\Sigma^{s,e}, \Sigma^{0,e}, \Sigma^{0,i}) \quad s.t \mid \Sigma^{s,e} \mid + \mid \Sigma^{0,e} \mid = n$ and $\mid \Sigma^{s,e} \mid > 0$ and $\mid \Sigma^{0,i} \mid = m$
  The union set of inequalities for $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$ is the one found in (4.8).
  In order to verify the inequalities, different combinations of $\boldsymbol{\sigma}$ must be taken into account:

– *If* $\sigma_k = s$
  For both Set A and Set B, the inequalities are verified thanks to $(Cond.\ 1A)$ and $(Cond.\ 1B)$

as:

$$\begin{aligned} u_j > -|W|_{j,k}m_k > \sum_{\substack{\sigma_t=s \\ t\in\{1,\dots,n\} \\ t\neq k}} (-|W|_{j,t}m_t) - |W|_{j,k}m_k \quad &Set\ A \\ u_j > 0 > \sum_{\substack{\sigma_t=s \\ t\in\{1,\dots,n\} \\ t\neq k}} (-|W|_{j,t}m_t) - |W|_{j,k}m_k \quad &Set\ B \end{aligned} \tag{6.11}$$

– *If* $\sigma_k = 0$
Again, for both Sets A and B the inequalities are verified because $(Cond.\ 4A)$ and $(Cond.\ 4B)$ holds:

$$\big(u_k > 0\big) > \sum_{\substack{\sigma_t=s \\ t\in\{1,\dots,n\}}} -|W|_{k,t}m_t \tag{6.12}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{0,i} \mid + \mid \Sigma^{s,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{0,e} \mid = n$
The union of inequalities that $\boldsymbol{u}$ must verify are the ones found in (4.9). A different approach will be considered for each $\boldsymbol{\sigma}$:

– *If* $\exists \sigma_i = s$ *such that* $i \in \{n+1, \dots, m+n\}$ *and* $i \neq j$
For both sets A and B, the inequalities are verified for $(Cond.\ 3A)$ and $(Cond.\ 3B)$, respectively, because:

$$u_i < -\sum_{e\in 1,\dots,n} |W|_{i,e}m_e < 0 < m_i + \sum_{\substack{t\in\{n+1,\dots,n+m\} \\ \sigma_t=s}} |W|_{i,t}m_t \tag{6.13}$$

– *If* $\sigma_j = s$ *and* $\forall i \in \{n+1, \dots, m+n\}$ *such that* $i \neq j$ *then* $\sigma_i = 0$
For both sets A and B, one inequality is verified by the superior bounds on u_j defined by $(Cond.\ 1A)$ and $(Cond.\ 1B)$. That is because:

$$u_j < -|W|_{j,k}m_k + (1 + |W|_{j,j})m_j < (1 + |W|_{j,j})m_j \tag{6.14}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{0,e} \mid + \mid \Sigma^{s,e} \mid = n$ and $s.t \mid \Sigma^{0,i} \mid + \mid \Sigma^{s,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{s,e} \mid > 0$
The union set of inequalities for the equilibrium point not to fall into its corresponding region are the ones in (4.10).
For each specific $\boldsymbol{\sigma}$ the inequalities will be verified using different conditions:

– *If there* $\exists \sigma_i = s$ *such that* $i \in \{n+1, \dots, m+n\}$ *and* $i \neq j$
For both Sets A and B, the inequalities are verified again by $(Cond.\ 3A)$ and $(Cond.\ 3B)$

because :

$$\Big(u_i < -\sum_{e\in\{1,\dots,n\}} |W|_{i,e}m_e\Big) < (1+|W|_{i,i})m_i - \sum_{\substack{\sigma_e=s\\ e\in\{1,\dots,n\}}} |W|_{i,e}m_e + \sum_{\substack{\sigma_r=s\\ r\in\{n+1,\dots,n+m\}\\ r\neq i}} |W|_{i,r}m_r \tag{6.15}$$

– *If $\sigma_j = s$ and $\forall i \in \{1,\dots,n\}$ such that $i \neq k$ then $\sigma_i = 0$ and $\sigma_k = s$*
For both Sets A and B, the inequalities are verified by $(Cond.\ 1A)$ and $(Cond.\ 1B)$ as:

$$u_j < \underbrace{-|W|_{j,k}m_k + (1+|W|_{j,j})m_j)}_{\sigma_k=s\quad\sigma_j=s} < -|W|_{j,k}m_k + (1+|W|_{j,j})m_j) + \sum_{\substack{\sigma_i=s\\ i\in\{n+1,\dots n+m,\}}} |W|_{j,i}m_i \tag{6.16}$$

– *If $\sigma_j = s$ and $\exists e \in \{1,\dots,n\}$ such that $\sigma_e = s$ and $e \neq k$*
Then, for both sets A and B, using $(Cond.\ 2A)$ and $(Cond.\ 2B)$ one has:

$$\begin{aligned} u_e &< m_e + \sum_{g\in\{1,\dots,n\}} \big(-|W|_{e,g}m_g\big) + |W|_{e,j}m_j \\ &< +|W|_{e,j}m_j + (-|W|_{e,e}+1)m_e + \sum_{\substack{g\in\{1,\dots,n\}\\ g\neq e\\ \sigma_g=s}} \big(-|W|_{e,g}m_g\big) + \sum_{\substack{\sigma_i=s\\ i\in\{n+1,\dots n+m,\}}} |W|_{j,i}m_i \end{aligned} \tag{6.17}$$

- $\boldsymbol{\sigma} = (0, \Sigma^{0,i}, \Sigma^{l,i})$ $\quad s.t \mid \Sigma^{l,i} \mid + \mid \Sigma^{0,i} \mid = m$ and $\mid \Sigma^{l,i} \mid > 0$
$\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ $\quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$
$\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$ $\quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$ and that $\mid \Sigma^{0,e} \mid + \mid \Sigma^{s,e} \mid = n$

The generalized inequalities that describes the valid inputs for these regions are the one in (4.11).
As before, we will treat the regions separately:

– $\boldsymbol{\sigma} = (\mathbf{0}, \Sigma^{0,i}, \Sigma^{l,i})$ and $\boldsymbol{\sigma} = (\mathbf{0}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$
This is the simplest case.
For set A, the inequalities will be verified as a consequence of lemma (5.1.9).
For the Set B, lemma (5.1.8), will guarantee that the inequalities are satisfied for every $\Sigma^{l,i}$ such that $|\Sigma^{l,i}| \geq 2$.
For $|\Sigma^{l,i}| = 1$ there are three approaches:

  * $\sigma_j = l$ and $\nexists i \in \{n+1,\dots,n+m\}$ such that $\sigma_i = s$
  The inequalities will be verified by $(cond.\ 4B)$ as:

$$u_k > \frac{|W|_{k,j}}{|W|_{j,j}+1} u_j \tag{6.18}$$

* $\sigma_j = l$ and $\exists i \in \{n+1,\dots,n+m\}$ such that $\sigma_i = s$
  The inequalities will be verified by (*cond.* 2*B*) as:

$$u_j < \min_{\substack{i \neq k,j \\ i \in \{n+1,\dots,m+n\}}} (|W|_{j,i} m_i) < \sum_{\substack{i \neq j \\ \Sigma^{s,i}_{i,i}=1}} |W|_{j,i} m_i \tag{6.19}$$

* $\sigma_j \neq l$
  Following the proof of (5.1.9) the inequalities will be verified as $\forall i \in \{n+1,\dots,m+n\}$ and $i \neq j$ then

$$u_i < 0 < \sum_{\substack{i \in \{n+1,\dots,n+m\} \\ i \neq j \\ \sigma_i = s}} |W|_{j,i} m_i \tag{6.20}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i})$
  If $| \Sigma^{l,i} | \geq 2$ then, both for set A and set B we can use lemma (5.2.7) to proof that the equilibrium point for these regions will fall outside them.
  For the regions where $| \Sigma^{l,i} | = 1$ multiple cases cases must be taken into account:

  * *It there $\exists \sigma_i = l$ such that $i \neq j$*
    Then for both sets, the inequalities that will be verified are the linear inequalities as:

$$u_i < -\sum_{e \in \{1,\dots,n\}} |W|_{i,e} m_e < -\sum_{\substack{e \in \{1,\dots,n\} \\ \sigma_e = s}} |W|_{i,e} m_e - \sum_{\substack{e \in \{n+1,\dots,n+m\} \\ 'e \neq i \\ \sigma_e = s}} |W|_{i,e} m_e \tag{6.21}$$

  * *If $\sigma_j = l$ and there $\exists \sigma_r = s$ such that $r \neq j$ and $r \in \{n+1,\dots,n+m\}$*
    The $r-th$ node will always evolve towards negative saturation as $u_r < -\sum_{e \in \{1,\dots,n\}} |W|_{r,e} m_e$, so the equilibrium point of these region will fall outside it.

  * *If $\sigma_j = l$ and there $\exists \sigma_r = s$ such that $r \neq k$ and $r \in \{1,\dots,n\}$ and $\forall i \in \{n+1,\dots,n+m\}$ such that $i \neq j$ $\sigma_i = 0$*
    The inequalities that are going to be verified are the ones with the forms of (*in.* 1) as :

$$u_e < \frac{(|W|_{e,j} u_j - \left[(|\boldsymbol{W}|_{e,1:n} - \boldsymbol{I}_n)(|W|_{j,j} + 1) - |W|_{e,j} |\boldsymbol{W}|_{j,1:n}\right]^{+} \boldsymbol{m}_e)}{(|W|_{j,j} + 1)} \tag{6.22}$$

  * *If $\sigma_j = l$ and $\sigma_k = s$ and $\sigma_r = \{0, s\}$ if $r \in \{n+1,\dots,m+n\}$ and $r \neq j$*
    It will be verified by either inequality (*in.* 0) or (*in.* 3) which they will have a form like the following

$$u_k < (1 - |W|_{k,k} + \frac{|W|_{k,j}|W|_{j,k}}{|W|_{j,j}+1}) m_k + \frac{|W|_{k,j}}{|W|_{j,j}+1} u_j + \sum_{t \neq j,k \in S} \left(|W|_{t,t} - \frac{|W|_{t,j}|W|_{j,t}}{|W|_{j,j}+1}\right)$$
$$u_j < -|W|_{j,k} m_k + \sum_{t \neq j, 1 \in S} |W|_{j,t} m_t$$

By inspection one of them will be always satisfied applying *(Sufficient condition 3)* and ($Cond.\ 4A$) and ($Cond.\ 4B$).
Note that, as the inhibitory nodes are manually saturated and they always evolve towards negative saturation, *(Sufficient condition 3)* is not necessary, not even strictly used. However, for the later algorithmic approach it will be used to relax the bound on the inputs of the inhibitory nodes.

$\square$

Corollary 6.2.2. ***(Sufficient conditions for the possible existence of non degenerate oscillations in a 1-excitatory - m-inhibitory network with algorithm based approach for u).***
For a system verifying the same set of sufficient conditions for $\boldsymbol{W}$ and $\boldsymbol{m}$ as in (6.8). It is possible to use the following algorithm to find an input $\boldsymbol{u}$ for which the system will have no stable equilibria and for the which the system will not necessarily oscillate in a non-degenerate mode.

Algorithm

Let for set A the initial conditions $\boldsymbol{u}_A^0$ be:

$$u_j \in \Big(\big(\tfrac{(|W|_{k,k}-1)(|W|_{j,j}+1)}{|W|_{k,j}} - |W|_{j,k}\big)m_k, \min(0, -|W|_{j,k}m_k + (1+|W|_{j,j})m_j)\Big) \qquad (Cond.\ 1A)$$

for $i \in \{n+1 \dots n+m\}$ **do**

  case $i \neq j$ **do**

$$u_i < \min\Big(\sum_{t\in\{1,\dots,n\}} -|W|_{i,t}m_t + \sum_{\substack{t\in\{n+1,\dots,n+m\}\\ t\neq i}} |W|_{i,t}m_i, \; -|W|_{i,k}m_k + (1+|W|_{i,i})m_i\Big) \qquad (Cond.\ 2A)$$

  end

end

for $e \in \{1 \dots n\}$ **do**

  case $e = k$ **do**

$$u_k \in \Big(0, (1 - |W|_{k,k} + \tfrac{|W|_{k,j}|W|_{j,k}}{1+|W|_{j,j}})m_k + \tfrac{|W|_{k,j}}{(1+|W|_{j,j})}u_j\Big) \qquad (Cond.\ 4A)$$

  case $e \neq k$ **do**

$$u_e < \min\Big(m_e + \sum_{g\in\{1,\dots,n\}} \big(-|W|_{e,g}m_g\big) + \min_{i\in\{n+1,\dots,n+m\}} |W|_{e,i}m_i, \qquad (Cond.\ 3A)$$
$$\tfrac{|W|_{e,j}}{|W|_{j,j}+1}(u_j + |W_{j,e}|m_e) + (1-|W|_{e,e})m_e - \textstyle\sum_{k\neq e,j}\big[W_{e,k} - \tfrac{|W|_{e,j}W_{j,k}}{|W|_{j,j}+1}\big]^+ m_k\Big)$$

  end

end

Let for set B the initial conditions $\boldsymbol{u}_B^0$ be:

$$u_j \in \Big(0, \min\big(-|W|_{j,k}m_k + (1+|W|_{j,j})m_j, \min_{\substack{i\in\{n+1,\dots,n+m\}\\ i\neq j}} (|W|_{j,i}m_i)\big)\Big) \qquad (Cond.\ 1B)$$

for $i \in \{n+1 \dots n+1\}$ **do**

  case $i \neq j$ **do**

$$u_i < \min\Big(\sum_{t\in\{1,\dots,n\}} -|W|_{i,t}m_t + \sum_{\substack{t\in\{n+1,\dots,n+m\}\\ t\neq i}} |W|_{i,t}m_i, \; -|W|_{i,k}m_k + (1+|W|_{i,i})m_i\Big) \qquad (Cond.\ 2B)$$

  end

end

for $e \in \{1 \dots n\}$ **do**

  case $e = k$ **do**

$$u_k \in \Big(\tfrac{|W|_{k,j}}{(1+|W|_{j,j})}u_j, (1 - |W|_{k,k} + \tfrac{|W|_{k,j}|W|_{j,k}}{1+|W|_{j,j}})m_k + \tfrac{|W|_{k,j}}{(1+|W|_{j,j})}u_j\Big) \qquad (Cond.\ 4B)$$

  case $e \neq k$ **do**

$$u_e < \min\Big(m_e + \sum_{g\in\{1,\dots,n\}} \big(-|W|_{e,g}m_g\big) + \min_{i\in\{n+1,\dots,n+m\}} |W|_{e,i}m_i, \qquad (Cond.\ 3B)$$
$$\tfrac{|W|_{e,j}}{|W|_{j,j}+1}(u_j + |W_{j,e}|m_e) + (1-|W|_{e,e})m_e - \textstyle\sum_{k\neq e,j}\big[W_{e,k} - \tfrac{|W|_{e,j}W_{j,k}}{|W|_{j,j}+1}\big]^+ m_k\Big)$$

  end

end

Let σ be a permutation of $\{n+1,\ldots,m+n\}$ such that $\sigma(j)=1$, so $P=(j,\sigma_2,\sigma_3,\ldots,\sigma_{m+1})$.
for $t=P(1):P(m)$ **do**
  for $w=P(t+1):P(m)$ **do**
    Let $T_{t,w}$ be the set of nodes $N\backslash\{i,j\}$.
    Let $S\subseteq T$.
    Let $O_S=\sum_{s\in S} O_s$
    forall S **do**
      $v=\Pi_{t,w}(\boldsymbol{u}-O_S)=[u_t-(O_S)_t,u_w-(O_s)_w]$
      if $v_t>0$ *and* $v_w>0$ **then**
        $M=\infty$
        $IW=(\boldsymbol{I}-\boldsymbol{W})$
        for $r=P(t):P(m)$ **do**
          $M=\min\big(M,\dfrac{IW_{w,r}}{IW_{t,r}}\big)$
        end
        if $\dfrac{v_w}{v_t}>M$ **then**
          $v_w=v_t\cdot K,\qquad K\in(0,M)$
        end
      end
      else if $v_t<0$ *and* $v_w>0$ **then**
        $v_w\in(-\infty,0)$
      end
      $u_w=v_w+(O_S)_w$
    end
  end
end

Proof. In order to proof that the algorithm will output an input $\boldsymbol{u}$ for which the system will have no stable equilibrium, an heuristic proof will be conducted, checking that for every possible $\boldsymbol{\sigma}$ the equilibrium candidate will not fall in its corresponding zone.

- $\boldsymbol{\sigma}=(0,0,\ldots,0)$
  The union set of inequalities that must be verified for this $\boldsymbol{\sigma}$ are the ones of found in (4.7). For both sets of possible initial conditions, A and B, the inequalities will be immediately verified by $u_k>0$ because of ($cond.\ 4A$) and ($cond.\ 4B$) .

- $\boldsymbol{\sigma}=(\Sigma^{s,e},\Sigma^{0,e},\Sigma^{0,i})\qquad s.t\mid\Sigma^{s,e}\mid+\mid\Sigma^{0,e}\mid=n$ and $\mid\Sigma^{s,e}\mid>0$ and $\mid\Sigma^{0,i}\mid=m$
  The union set of inequalities for $\boldsymbol{u}\in\mathbb{R}^N\backslash Y$ for this configuration of $\boldsymbol{\sigma}$ has the form of (4.8). In order to verify the inequalities, different combinations of $\boldsymbol{\sigma}$ must be taken into account:

  - *If $\sigma_k=s$*
    For both Set A ans Set B, the inequalities are verified thanks to ($Cond.\ 1A$) and ($Cond.\ 1B$) as:

$$u_j>-|W|_{j,k}m_k>\sum_{\substack{\sigma_t=s\\ t\in\{1,\ldots,n\}}}-|W|_{j,t}m_t \tag{6.23}$$

– *If* $\sigma_k = 0$

Again, for both Sets A and B the inequalities are verified because $(Cond.\ 4A)$ and $(Cond.\ 4B)$ holds:

$$u_k > 0 > \sum_{\substack{\sigma_t = s \\ t \in \{1,\ldots,n\}}} -|W|_{k,t} m_t \tag{6.24}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{0,i}, \Sigma^{s,i})$ $\quad s.t \mid \Sigma^{0,i} \mid + \mid \Sigma^{s,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{0,e} \mid = n$

  The union set of inequalities for the equilibrium candidate not to fall into the region $\boldsymbol{x}^* \in \mathbb{R}^N \backslash \Omega_{\boldsymbol{\sigma}}$ have the form of (4.9).

  The verification of the inequalities will be different for each $\boldsymbol{\sigma}$:

– *If* $\exists \sigma_i = s$ *such that* $i \in \{n+1, \ldots, m+n\}$ *and* $i \neq j$

For both sets A and B, the inequalities are verified for $(Cond.\ 1A)$ and $(Cond.\ 1B)$ respectively because:

$$u_i < -|W|_{i,k} m_k + (1 + |W|_{i,i}) m_i < + \sum_{\substack{t \in \{n+1,\ldots,n+m\} \\ \sigma_t = s \\ t \neq i}} |W|_{i,t} m_t + (1 + |W|_{i,i}) m_i \tag{6.25}$$

– *If* $\sigma_j = s$ *and* $\forall i \in \{n+1, \ldots, m+n\}$ *such that* $i \neq j$ *then* $\sigma_i = 0$

For both sets A and B, the inequalities are verified by the superior bounds defined by $(Cond.\ 1A)$ and $(Cond.\ 1B)$. That is because:

$$u_j < -|W|_{j,k} m_k + (1 + |W|_{j,j}) m_j < (1 + |W|_{j,j}) m_j \tag{6.26}$$

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{0,i}, \Sigma^{s,i})$ $\quad s.t \mid \Sigma^{0,e} \mid + \mid \Sigma^{s,e} \mid = n$ and $s.t \mid \Sigma^{0,i} \mid + \mid \Sigma^{s,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{s,e} \mid > 0$

  The union set of inequalities for the equilibrium points not to fall into its corresponding region are the ones displayed in (4.10).

  For each specific $\boldsymbol{\sigma}$ the inequalities will be verified using different conditions:

– *If* $\exists e \in \{1, \ldots, n\}$ *such that* $\sigma_e = s$ *and* $e \neq k$

Then, for both sets A and B, using $(Cond.\ 3A)$ and $(Cond.\ 3B)$ one has:

$$\begin{aligned} u_e &< m_e + \sum_{g \in \{1,\ldots,n\}} \big(-|W|_{e,g} m_g \big) + \min_{i \in \{n+1,\ldots,n+m\}} |W|_{e,i} m_i \\ &< +|W|_{e,j} m_j + (-|W|_{e,e} + 1) m_e + \sum_{\substack{g \in \{1,\ldots,n\} \\ g \neq e \\ \sigma_g = s}} \big(-|W|_{e,g} m_g \big) \end{aligned} \tag{6.27}$$

– *If* $\sigma_k = s$ *and* $\forall e \in \{1,\dots,n\}$ *such that* $e \neq k$ *then* $\sigma_e = 0$ *and* $\sigma_i = s$ *for some* $i \in \{n+1,\dots,n+m\}$ *then* $\sigma_i = s$

For both Sets A and B, the inequalities are verified by ($Cond.\ 1A$) and ($Cond.\ 1B$) or ($Cond.\ 2A$) and ($Cond.\ 2B$) as:

$$u_i < \underbrace{-|W|_{i,k}m_k + (1+|W|_{i,i})m_i)}_{\sigma_k=s \quad \sigma_i=s} < -|W|_{i,k}m_k + (1+|W|_{i,i})m_i) + \sum_{\substack{\sigma_t=s \\ t\in\{n+1,\dots n+m,\} \\ t\neq i}} |W|_{i,t}m_t \tag{6.28}$$

- $\boldsymbol{\sigma} = (0, \Sigma^{0,i}, \Sigma^{l,i}) \quad s.t \mid \Sigma^{l,i} \mid + \mid \Sigma^{0,i} \mid = m$ and $\mid \Sigma^{l,i} \mid > 0$
  $\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$
  $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$ and that $\mid \Sigma^{0,e} \mid + \mid \Sigma^{s,e} \mid = n$

The generalized inequalities that describes the valid inputs for these regions are (4.11).

We will check that all the hypothesis for the chain lemma, with displaced origins are verified, and so we will be able to affirm that no equilibrium point is contained in its corresponding regions for the $\boldsymbol{\sigma}$ of this concert form.

One can see that the algorithm presented it is equivalent to impose the chain condition with displaced origin (5.2.4) to a concrete permutation of nodes. The hypothesis for the chain lemma with displaced origin (5.1.7) are always verified as $\forall i \neq j$ and $i \in \{n+1,\dots,n+m\}$ it holds that:

$$u_i < \sum_{t\in\{1,\dots,n\}} -|W|_{i,t}m_t + \sum_{\substack{t\in\{n+1,\dots,n+m\} \\ t\neq i}} |W|_{i,t}m_i \tag{6.29}$$

If for the regions where $|\Sigma^{l,i}| = 1$ and $\sigma_j = l$ the inequalities are hold then for the corollary (5.2.6) we can affirm that all the equilibrium points of the regions where there is at least one inhibitory node in linear state will fall outside its corresponding zone.

Let's check it:

– $|\Sigma^{l,i}| = 1$ and $\sigma_j = l$

Let O_k be the origin when the k node is saturated and O_K the origin steering from and arbitrary set of excitatory nodes in positive saturation.

We have to consider different possibles:

  * $O_K = 0$ *and* $\forall i \in \{n+1,\dots,n+m\}$ *such that* $i \neq j$ *then* $\sigma_i = 0$

    The inequalities of this region that will be verified is:

$$u_k > \frac{|W|_{k,j}}{|W|_{j,j}+1}u_j \tag{6.30}$$

    In set B the condition is explicitly required. In Set A it is trivially verified as $u_j < 0$ while $u_k > 0$.

* $O_K = O_k$ *and* $\forall i \in \{n+1,\ldots,n+m\}$ *such that* $i \neq j$ *that* $\sigma_i = 0$
  The inequalities of this region that will be verified is:

$$u_k < (1 - |W|_{k,k} + \frac{|W|_{k,j}|W|_{j,k}}{1+|W|_{j,j}})m_k + \frac{|W|_{k,j}}{(1+|W|_{j,j})}u_j \tag{6.31}$$

  Which is a condition explicitly asked by the two sets of initial conditions.

* $O_K = O_k$ and $\exists i \in \{n+1,\ldots,n+m\}$ such that $i \neq j$ that $\sigma_i = 0$

  It will be verified by either inequality ($in.$ 0) or ($in.$ 3) which they will have a form like the following

$$u_k < (1 - |W|_{k,k} + \frac{|W|_{k,j}|W|_{j,k}}{|W|_{j,j}+1})m_k + \frac{|W|_{k,j}}{|W|_{j,j}+1}u_j + \sum_{t\neq j,k\in S}\big(|W|_{t,t} - \frac{|W|_{t,j}|W|_{j,t}}{|W|_{j,j}+1})$$
$$u_j < -|W|_{j,k}m_k + \sum_{t\neq j,1\in S}|W|_{j,t}m_t$$

  By inspection one of them be always verified applying *(Sufficient condition 3)*.

* $O_K = \sum_{t\in T\subseteq\{1,\ldots,n\}} O_t$
  By construction there exists an excitatory node $e \neq k$ in positive saturation state. The linear inequality that will be verified is :

$$u_e < \sum_{t\in 1,\ldots,m+n}\big(-sg(|W|_{e,t})|W|_{e,t}(\sum_{r\in 1,\ldots,m+n}(K^l_{t,r}(\Sigma^l)_{rr}u_r + K^l_{t,r}(\Sigma^s)_{rr}m_r)\big) + m_e \tag{6.32}$$

  Because this equation is equivalent to the imposed one in both sets:

$$u_e < \frac{|W|_{e,j}}{|W|_{j,j}+1}(u_j + |W_{j,e}|m_e) + (1-|W|_{e,e})m_e - \sum_{k\neq e,j}\big[W_{e,k} - \frac{|W|_{e,j}W_{j,k}}{|W|_{j,j}+1}\big]^+ m_k \tag{6.33}$$

□

Example 6.2.3. *For set A, two examples of $2n - 2m$ network with lack of stable equilibrium due to an input $\boldsymbol{u}$ generated by the previously presented algorithm will be presented.*
First and verifying assumption (2) with sufficient conditions (6.2), is the following:

$$\boldsymbol{W} = \begin{bmatrix} 12.4036 & 58.6785 & -5.1830 & -7.0132 \\ 33.2460 & 20.7908 & -4.4956 & -8.8692 \\ 7.2938 & 10.5526 & -4.6291 & -11.0126 \\ 24.7474 & 26.5113 & -13.6973 & -3.1799 \end{bmatrix} \quad \boldsymbol{m} = \begin{bmatrix} 11.8350 \\ 9.4887 \\ 19.5364 \\ 30.0699 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} -480.9566 \\ 3.2951 \\ -51.3739 \\ -132.6742 \end{bmatrix}$$

For the verifying assumption (2) with sufficient conditions (6.3), is the following:

$$\boldsymbol{W} = \begin{bmatrix} 17.1351 & 12.7555 & -6.7266 & -14.5233 \\ 37.1763 & 9.9317 & -3.3270 & -2.7642 \\ 27.0026 & 14.9951 & -1.1319 & -11.1408 \\ 31.8569 & 53.6773 & -5.0182 & -4.8149 \end{bmatrix} \quad \boldsymbol{m} = \begin{bmatrix} 0.7977 \\ 35.5728 \\ 31.9278 \\ 21.0029 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 2.8468 \\ -289.4243 \\ -321.0666 \\ -17.8773 \end{bmatrix}$$

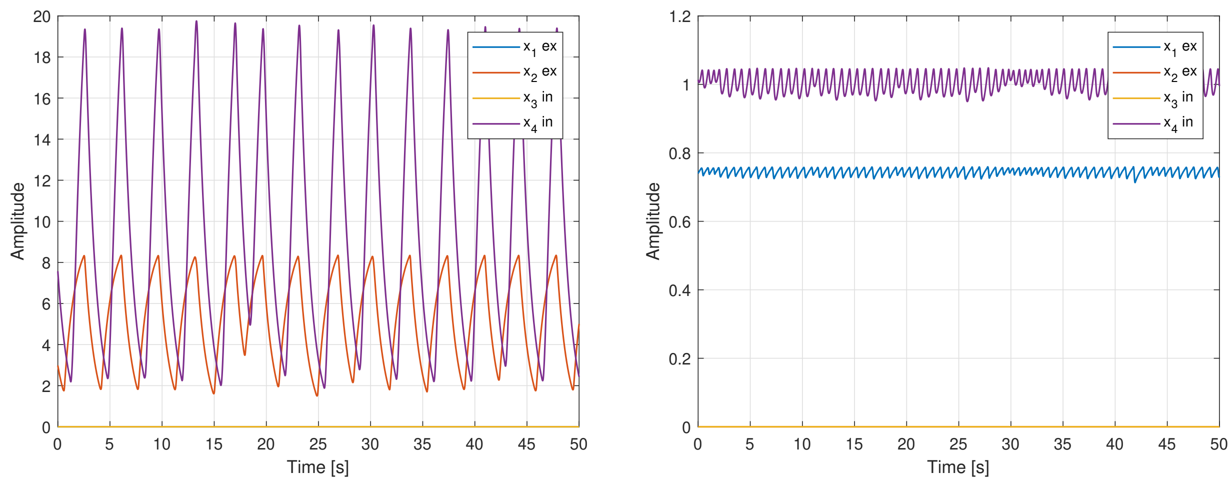


Figure 6.1: Temporal evolution of x_1 *ex*, x_2 *ex*, x_3 *in* and x_4 *in*. The system on the left verifies (6.2) and $k = 2$ and $j = 3$. The system on the rigth verifies (6.3) with $k = 1$ and $j = 4$

Both sets, even though not manually imposed tend to produce only two nodes oscillations (degenerate oscillation). In the subsequent sections we will compare the degenerate rate for this two different approaches for different n and m.

Example 6.2.4. *For set B, two examples of $2n - 2m$ network with lack of stable equilibrium due to an input $\boldsymbol{u}$ generated by the previously presented algorithm will be presented.*
First and verifying assumption (2) with sufficient conditions (6.2), is the following:

$$\boldsymbol{W} = \begin{bmatrix} 7.3780 & 27.0481 & -8.1789 & -9.0489 \\ 3.3510 & 11.9667 & -10.0072 & -2.5653 \\ 8.0968 & 17.1396 & -0.0525 & -8.3184 \\ 16.1142 & 30.1620 & -6.0649 & -2.9805 \end{bmatrix} \qquad \boldsymbol{m} = \begin{bmatrix} 2.2218 \\ 9.1639 \\ 37.0213 \\ 22.0661 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 104.5941 \\ -245.6105 \\ 5.2817 \\ -226.0109 \end{bmatrix}$$

For the verifying assumption (2) with sufficient conditions (6.3), is the following:

$$\boldsymbol{W} = \begin{bmatrix} 10.3003 & 18.6502 & -6.4151 & -0.9882 \\ 8.8789 & 9.4725 & -7.5235 & -6.5133 \\ 17.8521 & 7.8629 & -3.5068 & -20.5233 \\ 6.7837 & 24.7135 & -8.5086 & -2.8479 \end{bmatrix} \qquad \boldsymbol{m} = \begin{bmatrix} 15.8309 \\ 2.2815 \\ 4.0328 \\ 27.2482 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} -531.1022 \\ 7.4740 \\ 0.0369 \\ -129.5620 \end{bmatrix}$$

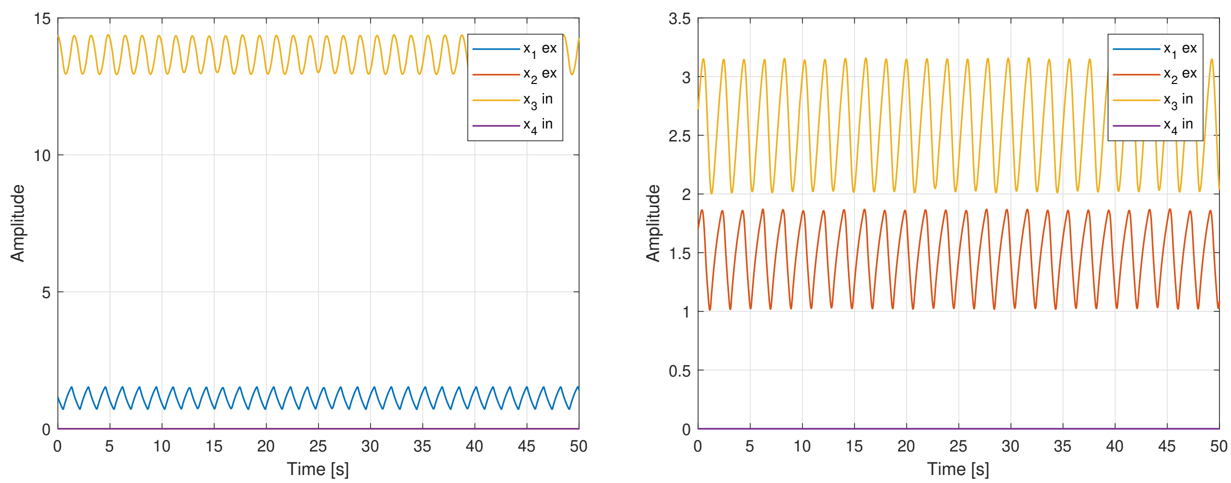


Figure 6.2: Temporal evolution of x_1 *ex*, x_2 *ex*, x_3 *in* and x_4 *in*. The system on the left verifies (6.2) and $k = 2$ and $j = 3$. The system on the right verifies (6.3) with $k = 1$ and $j = 4$

Both sets, even though not manually imposed tend to produce only two nodes oscillations (degenerate oscillation). In the subsequent sections we will compare the degenerate rate for this two different approaches for different n and m.

Theorem 6.2.5. ***(Sufficient conditions for the possible existence of non degenerate oscillations in a n-excitatory - m-inhibiroty network with a semi closed form for u, Set C).***
Consider a network with structure $(\boldsymbol{W},\boldsymbol{m})$ with n excitatory node and m inhibitory nodes. Assume that the assumption of instability (2) is verified.
Let $\mathbb{E}$ be the set of excitatory nodes.
Let $\mathbb{I}$ be the set of inhibitory nodes.
Let $\mathbb{J} \subseteq \mathbb{I}$ be the set of inhibitory nodes able to produce non-degenerate oscillations, and let $\mathbb{K} \subseteq \mathbb{E}$ be a set of excitatory nodes able to produce oscillations.
Let $A = (\boldsymbol{I} - \boldsymbol{W})$.
For the existence of a $\boldsymbol{u}$ for which the system will not have any stable equilibria is sufficient for the network to verify:

$$\begin{cases} \forall j \in \{n+1,\dots,m+n\} \text{ s.t. } j \in \mathbb{J} \text{ and } \forall k \in \{1,\dots,n\} \text{ s.t. } k \in \mathbb{K} & \\ \quad A_{j,e}m_e + A_{j,i}m_i < 0 \quad \forall e \in \mathbb{K} \text{ and } \forall i \in \mathbb{J}\backslash\{j\} & (Suff.\ Condition\ 1) \\ \quad A_{k,e}m_e + A_{k,i}m_i > 0 \quad \forall e \in \mathbb{K} \text{ and } \forall i \in \mathbb{J} & (Suff.\ Condition\ 2) \\ \quad \sum\limits_{e\in\mathbb{K}} A_{k,e}m_e + \sum\limits_{i\in\mathbb{J}} A_{k,i}m_i > 0 \quad \forall e \in \mathbb{K} & (Suff.\ Condition\ 3) \\ \forall l \text{ set of nodes s.t. } l \subseteq \mathbb{J} \text{ and } |l| \geq 1 & (Suff.\ Condition\ 4) \\ \quad \text{Let } \boldsymbol{M}^l = \boldsymbol{W}(\boldsymbol{I} - \Sigma^l\boldsymbol{W})^{-1} \text{ then :} & \\ \quad \forall R \subseteq \mathbb{K}, \text{ set of excitatory nodes } \exists r \in R \text{ such that:} & \\ \qquad \Big(diag(\boldsymbol{M}^l) + \big[\boldsymbol{M}^l - diag(\boldsymbol{M}^l)\big]^+)(\boldsymbol{I} - \Sigma^l)(\Sigma^R + \Sigma^{\mathbb{J}})\boldsymbol{m})\Big)_r < \boldsymbol{m}_r & \end{cases} \tag{6.34}$$

Proof. We will proof that it always exists a $\boldsymbol{u}$ in the set $\mathbb{U}$ for which each equilibrium point does not fall in its corresponding region.

$$\mathbb{U} \subset \mathbb{R}^{m+n} = \prod_i \mathbb{U}_i \text{ where } \mathbb{U}_i = \begin{cases} (0, \varepsilon_i) & If\ i \in \mathbb{K} & (Cond.\ 1) \\ (-\gamma_i, 0) & If\ i \in \mathbb{J} & (Cond.\ 2) \\ (-\infty, \sum\limits_{e\in\{1,\dots n\}} -|W|_{i,e}m_e) & If\ i \in \mathbb{I}\backslash\mathbb{J} & (Cond.\ 3) \\ (-\infty, \sum\limits_{t\in\{1,\dots n\}} -|W|_{i,t}m_t) & If\ i \in \mathbb{E}\backslash\mathbb{K} & (Cond.\ 4) \end{cases} \tag{6.35}$$

The proof will consists on considering each possible region $\boldsymbol{\sigma}$ and checking that the equilibrium does not fall into it if the $\boldsymbol{u}$ has that specific form, and giving the values of ε_i while doing so.

- $\boldsymbol{\sigma} = (0, 0, \dots, 0)$
  The equilibrium will not fall in this concrete region if $\exists\ u_i$ such that $u_i > 0$. It will be immediately verified by any of the $u_i > 0$ with $i \in \mathbb{K}$ as $(Cond.\ 1)$ provides.

- $\boldsymbol{\sigma} = (\Sigma^{s,e}, \Sigma^{0,e}, \Sigma^{s,i}, \Sigma^{0,i}) \qquad s.t \mid \Sigma^{s,e} \mid + \mid \Sigma^{0,e} \mid = n$ and $\mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$.
  The inequalities that describe the valid region are the ones of (4.10).
  We will distinguish several cases inside this one:

  - $\exists i \mathbb{E}\backslash\mathbb{K}$ *for which* $(\Sigma^s)_{i,i} = 1$
    This will be verified by condition ($Cond.$ 4) as, for some i such that $\sigma_i = s$, one would have:

$$u_i < \sum_{t\in\{1,\dots,n\}} (-|W|_{i,t} m_t) < \sum_{\substack{t\in\{1,\dots,n\}\\ \sigma_t=s}} -|W|_{i,t} m_t + \sum_{\substack{t\in\{n+1,\dots,n+m\}\\ \sigma_t=s}} |W|_{i,t} m_t \tag{6.36}$$

  - $\exists i \mathbb{I}\backslash\mathbb{J}$ *for which* $(\Sigma^s)_{i,i} = 1$
    The inequalities will be verified by condition ($Cond.$ 3) as, for some i such that $\sigma_i = s$, one would have:

$$u_i < \sum_{t\in\{1,\dots,n\}} (-|W|_{i,t} m_t) < \sum_{\substack{t\in\{1,\dots,n\}\\ \sigma_t=s}} -|W|_{i,t} m_t + \sum_{\substack{t\in\{n+1,\dots,n+m\}\\ \sigma_t=s}} |W|_{i,t} m_t \tag{6.37}$$

  - $\nexists i \in \mathbb{I}\backslash\mathbb{J}$ *or* $\mathbb{E}\backslash\mathbb{K}$ *for which* $(\Sigma^s)_{i,i} = 1$ *and* $\exists e \in \mathbb{K}$ *for which* $(\Sigma^s)_{e,e} = 1$ *and* $\nexists j \in \mathbb{J}$ *for which* $(\Sigma^s)_{j,j} = 1$
    This will be verified by ($Cond.$ 2) setting the bounds γ_j to be

$$\gamma_j < \min_{e\in\{1,\dots,n\}} (|W|_{j,e} m_e) \tag{6.38}$$

    Then it will hold that:

$$u_j > -\gamma_j > -\min_{e\in\{1,\dots,n\}} (|W|_{j,e} m_e) > \sum_{\substack{e\in\{1,\dots,n\}\\ \sigma_e=s}} -|W|_{j,e} m_e \tag{6.39}$$

  - $\nexists i \in \mathbb{I}\backslash\mathbb{J}$ *or* $\mathbb{E}\backslash\mathbb{K}$ *for which* $(\Sigma^s)_{i,i} = 1$ *and* $\nexists e \in \mathbb{K}$ *for which* $(\Sigma^s)_{e,e} = 1$ *and* $\exists j \in \mathbb{J}$ *for which* $(\Sigma^s)_{j,j} = 1$
    This will also be verified by ($Cond.$ 2) as $u_j < 0$ because it trivially holds that:

$$u_j < 0 < \sum_{\substack{i\in\{n+1,\dots,n+m\}\\ \sigma_i=s}} |W|_{j,i} m_i + m_i \tag{6.40}$$

  - $\nexists i \in \mathbb{I}\backslash\mathbb{J}$ *or* $\mathbb{E}\backslash\mathbb{K}$ *for which* $(\Sigma^s)_{i,i} = 1$ *and* $\exists e \in \mathbb{K}$ *for which* $(\Sigma^s)_{e,e} = 1$ *and* $\exists j \in \mathbb{J}$ *for which* $(\Sigma^s)_{j,j} = 1$
    For these concrete regions, the inequalities will be verified thanks to the combinations of conditions ($Cond.$ 1) and ($Cond.$ 2) with the *(Sufficient condition 1)* and *(Sufficient condition 2)*.
    For that, let's consider that $\exists j \in \mathbb{J}$ such that $(\Sigma^s)_{j,j} = 0$. We can proof by comparison that either the inequalities defined by this specific node j in negative saturation or the inequalities

defined by some node k in positive saturated state, will always be satisfied. We have:

$$\begin{aligned} u_j &> -\sum_{\substack{e\in\mathbb{K}\\ \sigma_e=s}} |W|_{j,e} m_e + \sum_{\substack{i\in\mathbb{J}\\ \sigma_i=s}} |W|_{j,i} m_i = \sum_{\substack{e\in\mathbb{K}\\ \sigma_e=s}} A_{j,e} m_e + \sum_{\substack{i\in\mathbb{J}\\ \sigma_i=s}} A_{j,i} m_i \quad (1)\\ u_k &< -\sum_{\substack{e\in\mathbb{K}\\ \sigma_e=s}} |W|_{k,e} m_e + \sum_{\substack{i\in\mathbb{J}\\ \sigma_i=s}} |W|_{k,i} m_i + m_k = \sum_{\substack{e\in\mathbb{K}\\ \sigma_e=s}} A_{k,e} m_e + \sum_{\substack{i\in\mathbb{J}\\ \sigma_i=s}} A_{k,i} m_i \quad (2) \end{aligned} \tag{6.41}$$

So it can be seen that if $|\Sigma^{s,i}| > |\Sigma^{s,e}|$ then inequalities (2) will be verified, for *(Sufficient condition 2)*.
If $|\Sigma^{s,i}| < |\Sigma^{s,e}|$ then inequalities (1) will be verified, for *(Sufficient condition 1)*.
If there doesn't exist any node j in negative saturation then it is *(Sufficient condition 3)* and the inequality (2) the one verified.

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t\ |\Sigma^{s,i}| + |\Sigma^{0,i}| + |\Sigma^{l,i}| = m$ and $|\Sigma^{0,e}| = n$
  The inequalities that will need to be verified are the ones exposed in (4.11) and all the sets will be verified because $u_i < 0\ \forall i \in \{n+1, \ldots, m+n\}$ as lemma (5.1.9) implies.

- $\boldsymbol{\sigma} = (\Sigma^{0,e}, \Sigma^{s,e}, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t\ |\Sigma^{s,i}| + |\Sigma^{0,i}| + |\Sigma^{l,i}| = m$ and $|\Sigma^{0,e}| + |\Sigma^{s,e}| = n$
  The inequalities that define a valid input are (4.11).
  In order to proof that all sets are verified let's introduce some notation. Let $\mathbb{I}_l$ refer to the set of all inhibitory nodes in linear state and let $\mathbb{J}_l$ refer to the nodes such that $j \in \mathbb{J}$ and j is in linear state. According to this notation, we will distinguish different cases:

– $\mathbb{I}_l \backslash \mathbb{J} \neq \emptyset$ *and/or* $\exists e \in \mathbb{E}\backslash\mathbb{K}$ *such that* $(\Sigma^s)_{e,e} = 1$
If $\exists i \in \mathbb{I}_l\backslash\mathbb{J}$ then the evolution of this $i-th$ node will always go to negative saturation, so the equilibrium point will be trivially out of the linear region.
If $\exists e \in \mathbb{E}\backslash\mathbb{K}$, the evolution of this $e-th$ node will also go towards negative saturation, implying that the equilibrium point will be trivially outside the positive saturation $e-th$ region.

– $\mathbb{I}_l \backslash \mathbb{J} = \emptyset$ *and* $\nexists e \in \mathbb{E}\backslash\mathbb{K}$ *such that* $(\Sigma^s)_{e,e} = 1$
Here we will consider the inequality $(in.0)$ of the set of inequalities shown in (**??**).
For every $k \in \mathbb{K}$ in positive saturation state these inequalities are:

$$\left((\boldsymbol{W}(\boldsymbol{I} - \Sigma^l \boldsymbol{W})^{-1}(\Sigma^l \boldsymbol{u} + \Sigma^s \boldsymbol{m})\right)_k + u_k < m_k \tag{6.42}$$

As we can considerate $u_j \in (-\gamma_j, 0)$ and $u_k \in (0, \varepsilon_k)$, with arbitrary small $|\gamma_j|$ and $|\varepsilon_j|$, then it is sufficient for having "room" for u_j and u_k that $\forall R \subseteq \mathbb{K}$ and $\forall S \subseteq \mathbb{J}\backslash\mathbb{J}_l$ $\exists r \in R$ such that:

$$\begin{gathered} \left((\boldsymbol{W}(\boldsymbol{I} - \Sigma^l \boldsymbol{W})^{-1}((\Sigma^R + \Sigma^S)\boldsymbol{m})\right)_k = \boldsymbol{M}^l((\Sigma^R + \Sigma^S)\boldsymbol{m}) = \\ \Big(\big(diag(\boldsymbol{M}^l) + \boldsymbol{M}^l - diag(\boldsymbol{M}^l)\big)((\Sigma^R + \Sigma^S)\boldsymbol{m})\Big)_r < m_r \end{gathered} \tag{6.43}$$

Let's fix a $k \in \mathbb{K}$, and suppose this node is in positive saturation while all the other $e \in \mathbb{K}$ remain in negative saturation. As S can be an arbitrary set not including any node in Σ^l.

Given that, one can see that the worst possible case will happen when:

$$\begin{aligned} &\Big(\big(diag(\boldsymbol{M}^l) + \boldsymbol{M}^l - diag(\boldsymbol{M}^l)\big)((\Sigma^k + \Sigma^S)\boldsymbol{m})\Big)_k \\ \leq &\Big(\big(diag(\boldsymbol{M}^l) + [\boldsymbol{M}^l - diag(\boldsymbol{M}^l)]^+\big)((\boldsymbol{I} - \Sigma^l)(\Sigma^k + \Sigma^{\mathbb{J}})\boldsymbol{m}))\Big)_k < m_k \end{aligned} \tag{6.44}$$

Furthermore, if we want this to be true for every $R \subseteq \mathbb{K}$ we must impose *(Sufficient condition 4)*, and it will always hold for at least one r such that $\Sigma^R_{r,r} = 1$ so:

$$\Big(\big(diag(\boldsymbol{M}^l) + [\boldsymbol{M}^l - diag(\boldsymbol{M}^l)]^+\big)((\boldsymbol{I} - \Sigma^l)(\Sigma^R + \Sigma^{\mathbb{J}})\boldsymbol{m})\Big)_r < m_r \tag{6.45}$$

Thus implies that there will always exists sufficiently small u_j with $j \in \mathbb{J}$ and a sufficiently small u_k with $k \in \mathbb{K}$ for which at least one inequality hold for every Σ^s and Σ^R.

Then, although other forms can be found, some possible boundaries for γ_j and ε_i are going to be proposed. So let $B^l_{R,t}$ be:

$$B^l_{R,t} = \big((diag(\boldsymbol{M}^l) + [M^l - diag(\boldsymbol{M}^l)]^+(\boldsymbol{I} - \Sigma^l)((\Sigma^R + \Sigma^{\mathbb{J}}))\boldsymbol{m}\big)_t \tag{6.46}$$

For the t that the inequality is verified ($< m_t$). Then let B^l_R be.

$$B^l_R = \min_t B^l_{R,t} \tag{6.47}$$

Note that, B^l_R is not empty because *(Sufficient condition 4)* will always guarantee that at least one t exists. Also, for that specific t^*, which is the minimum, we have that that $m_{t^*} - B^l_R > 0$. Furthermore, note that for $R = t$ then necessarily $t^* = t$ so every $k \in \mathbb{K}$ will be the minimum value at least once. Taking into account the inequality, and letting S be the set of positive saturated nodes such that $S \cap \mathbb{J}_l = \emptyset$ and $R = \mathbb{E} \cap S$ one has that:

$$\begin{aligned} u_k + \boldsymbol{M}^l(\Sigma^l\boldsymbol{u} + \Sigma^s\boldsymbol{m}) &< m_k \\ u_k &< m_k - \boldsymbol{M}^l(\Sigma^l\boldsymbol{u} + \Sigma^s\boldsymbol{m}) \end{aligned} \tag{6.48}$$

Taking into account that $u_j < 0$, a sufficient conditions for our values will be that:

$$u_k < m_k - B^l_R - ([\boldsymbol{M}^l]^-)\Sigma^l\boldsymbol{u} < m_k - \boldsymbol{M}^l(\Sigma^l\boldsymbol{u} + \Sigma^S\boldsymbol{m}) \tag{6.49}$$

So it is sufficient, for verifying all the inequalities that:

$$\begin{cases} \varepsilon_k = \min\Big(\sum\limits_{e\in\mathbb{K}} A_{k,e}m_e + \sum\limits_{i\in\mathbb{J}} A_{k,i}m_i, \big(\min\limits_{\substack{i\in\mathbb{J}\\ e\in\mathbb{K}}}(A_{k,e}m_e + A_{k,i}m_i\big), \big(\min\limits_{\substack{\forall l\subseteq\mathbb{J}\\ |l|\geq 1\\ \forall R\ s.t.\ t^*=k}} \dfrac{m_{t^*} - B^l_R}{\#[(\boldsymbol{M}^l\Sigma^l)^-_{t^*,:}] + 1}\big)\Big) & \forall k \in \mathbb{K} \\ \gamma_j = \min\Big(\big(\min\limits_{\substack{i\in\mathbb{J}\setminus\{j\}\\ e\in\mathbb{K}}}(A_{j,e}m_e + A_{j,i}m_i)\big), \big(\min\limits_{\substack{\forall l\subseteq\mathbb{J}\\ |l|\geq 1\\ \forall R}} \dfrac{m_{t^*} - B^l_R}{(\#[(\boldsymbol{M}^l\Sigma^l)^-_{t^*,:}] + 1)|\boldsymbol{M}^l_{t^*,j}|}\big)\Big) & \forall i \in \mathbb{J} \end{cases} \tag{6.50}$$

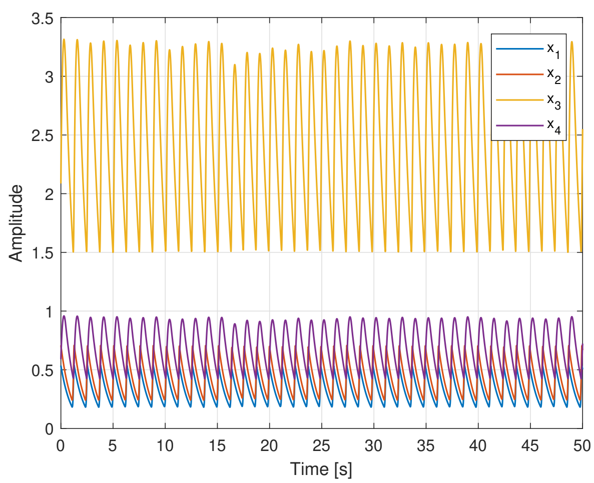
$\square$

Example 6.2.6. *For set C, two examples of networks with lack of stable equilibrium due to an input $\boldsymbol{u}$ generated by the previously presented algorithm will be presented.*
First a a $2n-2m$ network, which verifies assumption (2) with sufficient conditions (6.2):

$$\boldsymbol{W} = \begin{bmatrix} 4.0229 & 6.2981 & -11.7053 & -3.6931 \\ 8.4590 & 5.6747 & -12.0643 & -4.6839 \\ 23.8769 & 20.7008 & -0.0650 & -4.0205 \\ 5.9347 & 7.6329 & -0.0872 & -0.6337 \end{bmatrix} \qquad \boldsymbol{m} = \begin{bmatrix} 8.9027 \\ 13.5210 \\ 14.3854 \\ 29.7487 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 18.9833 \\ 19.5192 \\ -13.0923 \\ -4.5531 \end{bmatrix}$$

A $1n-2m$ network, which verifies assumption (2) with sufficient conditions (6.3), is the following:

$$\boldsymbol{W} = \begin{bmatrix} 6.7224 & -8.2818 & -12.7841 \\ 16.6541 & -1.4874 & -19.9469 \\ 22.1595 & -20.7662 & -2.1835 \end{bmatrix} \qquad \boldsymbol{m} = \begin{bmatrix} 23.6491 \\ 18.9782 \\ 16.1011 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 2.4844 \\ -0.5698 \\ -2.3293 \end{bmatrix}$$



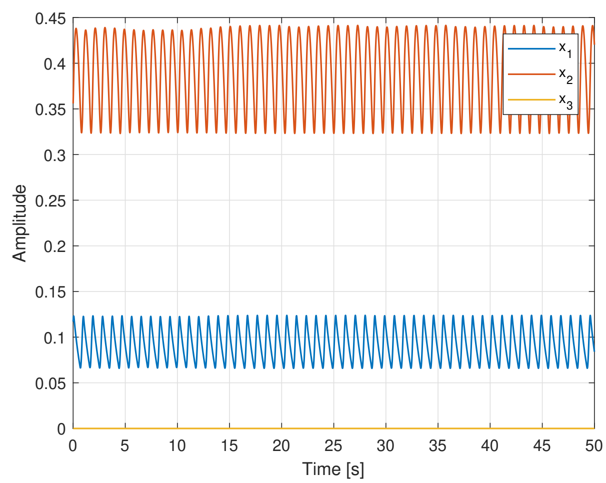


Figure 6.3: Temporal evolution of x_1 ex, x_2 ex, x_3 in and x_4 in, on the left, which verifies (6.2). Temporal evolution of x_1 ex, x_2 ex, x_3 in, on the right, which verifies (6.3).

Note that for the second example, it was numerically not possible to find a $\boldsymbol{W}$ $2n-2m$ under assumption (6.3) that would verify the sufficient conditions.
Differently as before, the oscillations are not produced only by two nodes. However this is not the usual pattern. In later sections it will be compared the degenerate rate for this first approach only, for different n and m.

6.3 Partial results for 1E- mI networks

Corollary 6.3.1. ***(Conditions for the existence of oscillations in a 1-excitatory - m-inhibiroty network).***
Consider the dynamics (2.7)-(2.8), with 1 excitatory node and m inhibiroty nodes, and assume $\boldsymbol{W}$ *verifies the same assumptions (4.1) as in Theorem I.1. For the system to not have any stable equilibria* [2]*, it is necessary that for every* Σ^s *exists at least one* u_i *that verifies one of these conditions:*

$$\begin{cases} \big((\boldsymbol{I}-\boldsymbol{W})\boldsymbol{\Sigma}^s\boldsymbol{m}\big)_i > \big(\boldsymbol{\Sigma}^s\boldsymbol{u}\big)_i \\ \big((\boldsymbol{I}-\boldsymbol{W})\boldsymbol{\Sigma}^s\boldsymbol{m}\big)_i < \big((\boldsymbol{I}-\boldsymbol{\Sigma}^s)\boldsymbol{u}\big)_i \end{cases} \tag{6.51}$$

and it is sufficient for the system that both $\boldsymbol{m}$ *and* $\boldsymbol{W}$ *verify the following conditions:*

$$\begin{cases} For\ some\ j \in \{2,\dots,m+1\}: \\ \quad (|W|_{1,1}-1)(1+|W|_{j,j}) < |W|_{j,1}|W|_{1,j} & (sufficient\ condition\ 1) \\ \quad (|W|_{1,1}-1)m_1 < |W|_{1,j}m_j \ \ or\ \ (|W|_{j,j}+1)m_j > |W|_{j,1}m_1 & (sufficient\ condition\ 2) \\ \quad \forall i \neq j,1 & (sufficient\ condition\ 3) \\ \quad\quad (|W|_{1,i}(|W|_{j,j}+1) - |W|_{1,j}|W|_{j,i} > 0)\ or\ (|W|_{j,i}m_i > (|W|_{j,j}+1)m_j) \end{cases} \tag{6.52}$$

For this set of sufficient conditions two different sets of conditions on $\boldsymbol{u}$ *will be given.*

- *Set A:*
  Let $\boldsymbol{u}$ *be such that:*

$$\begin{cases} for\ i \neq j\ and\ i \in \{2,\dots,m+1\} \\ \quad u_i < -|W|_{i,1}m_1 & (cond.\ 1A) \\ for\ j \\ \quad \left(\frac{(|W|_{1,1}-1)(|W|_{j,j}+1)}{|W|_{1,j}} - |W|_{j,1}\right)m_1 < u_j < \min\left(0, -|W|_{j,1}m_1 + (1+|W|_{j,j})m_j\right) & (cond.\ 2A) \\ for\ i=1 \\ \quad 0 < u_1^e < \left(1-|W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{1+|W|_{j,j}}\right)m_1 + \frac{|W|_{1,j}}{(1+|W|_{j,j})}u_j & (cond.\ 3A) \end{cases} \tag{6.53}$$

[2]The system will have unstable equilibrium for the switching regions where the excitatory node is in linear state and it will not have equilibrium, as the equilibrium point will fall outside of the switching region, for the other regions

- *Set B:*
  Let $\boldsymbol{u}$ be such that:

$$\begin{cases} for\ i \neq j\ and\ i \in \{2,\dots,m+1\} & \\ \quad u_i < -|W|_{i,1} m_1 & (cond.\ 1B) \\ for\ j & \\ \quad 0 < u_j < \min\left(-|W|_{j,1} m_1 + (1+|W|_{j,j}) m_j, \min\limits_{i \neq 1,j}(|W|_{j,i} m_i)\right) & (cond.\ 2B) \\ for\ i = 1 & \\ \quad \frac{|W|_{1,j}}{(1+|W|_{j,j})} u_j < u_1^e < (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{1+|W|_{j,j}}) m_1 + \frac{|W|_{1,j}}{(1+|W|_{j,j})} u_j & (cond.\ 3B) \end{cases} \tag{6.54}$$

Remark 6.3.2. *The constructed $\boldsymbol{u}$ used in the proof makes the oscillatory behavior of the E-mI system equivalent to E-I system, producing degenerate oscillations.*
Example 6.3.3. *An example of an input $\boldsymbol{u}$ which gives degenerate oscillations in a 1E-2I system, generated following the construction in the proof is the following:*

$$\boldsymbol{W} = \begin{bmatrix} 7.4075 & -8.5254 & -8.0797 \\ 2.9585 & -1.0396 & -9.1000 \\ 14.3509 & -5.2415 & -2.9803 \end{bmatrix} \qquad \boldsymbol{m} = \begin{bmatrix} 9.4185 \\ 8.8360 \\ 2.6957 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 3.9105 \\ -10.6096 \\ -135.2635 \end{bmatrix} \tag{6.55}$$

The temporal evolution is the following:

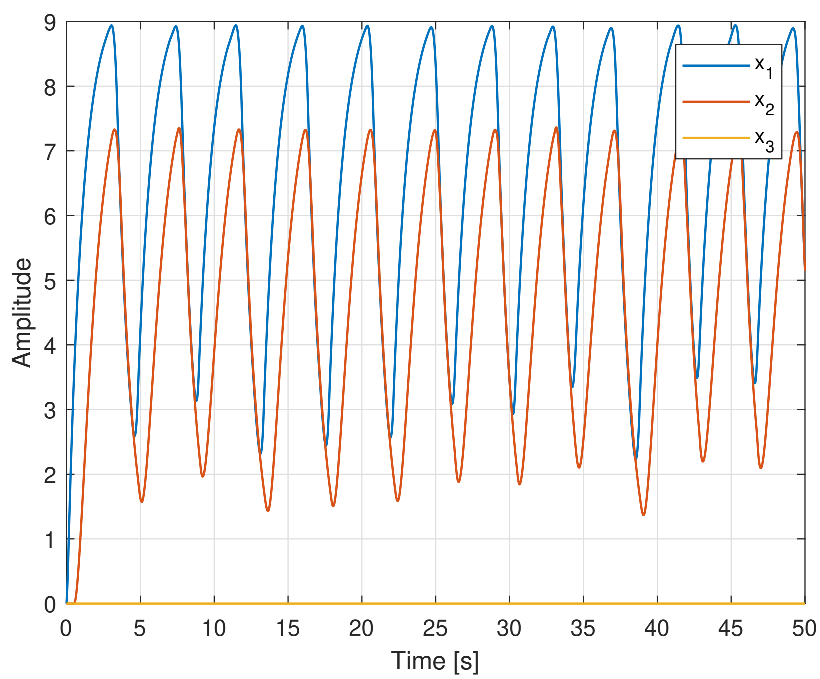


Figure 6.4: Temporal evolution of x_1, x_2, x_3 with x_1 being an excitatory node and both x_2 and x_3 being inhibitory nodes. For this system $j = 2$.

Corollary 6.3.4. ***(Sufficient conditions for the possible existence of non degenerate oscillations in a 1-excitatory - m-inhibiroty network with algorithm based approach for $\boldsymbol{u}$).***
For a system verifying the same set of sufficient conditions for $\boldsymbol{W}$ and $\boldsymbol{m}$ as in (6.52). It is possible to use the following algorithm to find an input $\boldsymbol{u}$ for which the system will have no stable equilibria and for the which the system will not necessarily oscillate in a non-degenerate mode.

Algorithm for set A and set B

Let $\boldsymbol{u}_A^0$ *be:*
$u_j \in \Big(\big(\frac{(|W|_{1,1}-1)(|W|_{j,j}+1)}{|W|_{1,j}} - |W|_{j,1}\big)m_1, \min(0, -|W|_{j,1}m_1 + (1+|W|_{j,j})m_j) \Big)$ (*Cond.* 2*A*)
for $i \in \{1 \dots N\}$ **do**
  case $i = 1$ **do**
    $u_1 \in \big(0, (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{1+|W|_{j,j}})m_1 + \frac{|W|_{1,j}}{(1+|W|_{j,j})}u_j\big)$ (*Cond.* 3*A*)
  case $i \neq 1$ *and* $i \neq j$ **do**
    $u_i < \min\Big(-|W|_{i,1}m_1 + (1+|W|_{i,i})m_i, \min\limits_{k\neq 1,i}(|W|_{i,k}m_k)\Big)$ (*Cond.* 1*A*)
  end
end
Let $\boldsymbol{u}_B^0$ *be:*
$u_j \in \Big(0, \min\big(-|W|_{j,1}m_1 + (1+|W|_{j,j})m_j, \min\limits_{i\neq 1,j}(|W|_{j,i}m_i)\big)\Big)$ (*Cond.* 2*B*)
for $i \in \{1 \dots N\}$ **do**
  case $i = 1$ **do**
    $u_1 \in \big(\frac{|W|_{1,j}}{1+|W|_{j,j}}u_j, (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{1+|W|_{j,j}})m_1 + \frac{|W|_{1,j}}{(1+|W|_{j,j})}u_j\big)$ (*Cond.* 3*B*)
  case $i \neq 1$ *and* $i \neq j$ **do**
    $u_i < \min\Big(-|W|_{i,1}m_1 + (1+|W|_{i,i})m_i, \min\limits_{k\neq 1,i}(|W|_{i,k}m_k)\Big)$ (*Cond.* 1*B*)
  end
end

Let σ *be a permutation of* $\{2, \dots, m+1\}$ *such that* $\sigma(j) = 1$, *so* $P = (j, \sigma_2, \sigma_3, \dots, \sigma_{m+1})$.
for $O_s = 0$ *and* $O_s = O_1$ **do**
  for $t = P(1) : P(m)$ **do**
    for $w = P(t+1) : P(m)$ **do**
      $v = \Pi_{t,w}(\boldsymbol{u} - O_1) = [u_t - (O_1)_t, u_w - (O_1)_w]$
      if $v_t > 0$ *and* $v_w > 0$ **then**
        $M = \infty$
        $IW = (\boldsymbol{I} - \boldsymbol{W})$
        for $r = P(t) : P(m)$ **do**
          $M = \min\big(M, \frac{IW_{w,r}}{IW_{t,r}}\big)$ (*Cond.* 4)
        end
        if $\frac{v_w}{v_t} > M$ **then**
          $v_w = v_t \cdot K, \quad K \in (0, M)$
        end
      end
      else if $v_t < 0$ *and* $v_w > 0$ **then**
        $v_w \in (-\infty, 0)$
      end
      $u_w = v_w + (O_1)_w$
    end
  end
end

Remark 6.3.5. ***(Efficiency of the algorithm).***
The algorithm will have an asymptotic cost of $\Theta(2^n) * \Theta(m^2)$ *in time and* $\Theta(N^2)$ *in memory.*
Example 6.3.6. *Finally, an example for a system verifying the sufficient conditions where non-degenerate oscillations are found for a concrete input* $\boldsymbol{u}$ *is going to be given. With these it will be proven the possible existence of non-degenerate oscillation. The system is the following:*

$$\boldsymbol{W} = \begin{bmatrix} 11.7164 & -0.9469 & -9.2096 & -9.9762 \\ 0.5880 & -3.8596 & -7.6311 & -3.2357 \\ 2.5245 & -3.2044 & -1.0570 & -2.0053 \\ 3.1832 & -4.4333 & -3.1199 & -2.6370 \end{bmatrix} \quad \boldsymbol{m} = \begin{bmatrix} 5.7266 \\ 5.3833 \\ 9.4078 \\ 12.9852 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 15.5589 \\ 3.4224 \\ 2.9341 \\ 3.9593 \end{bmatrix} \tag{6.56}$$

So the temporal evolution of the system is :

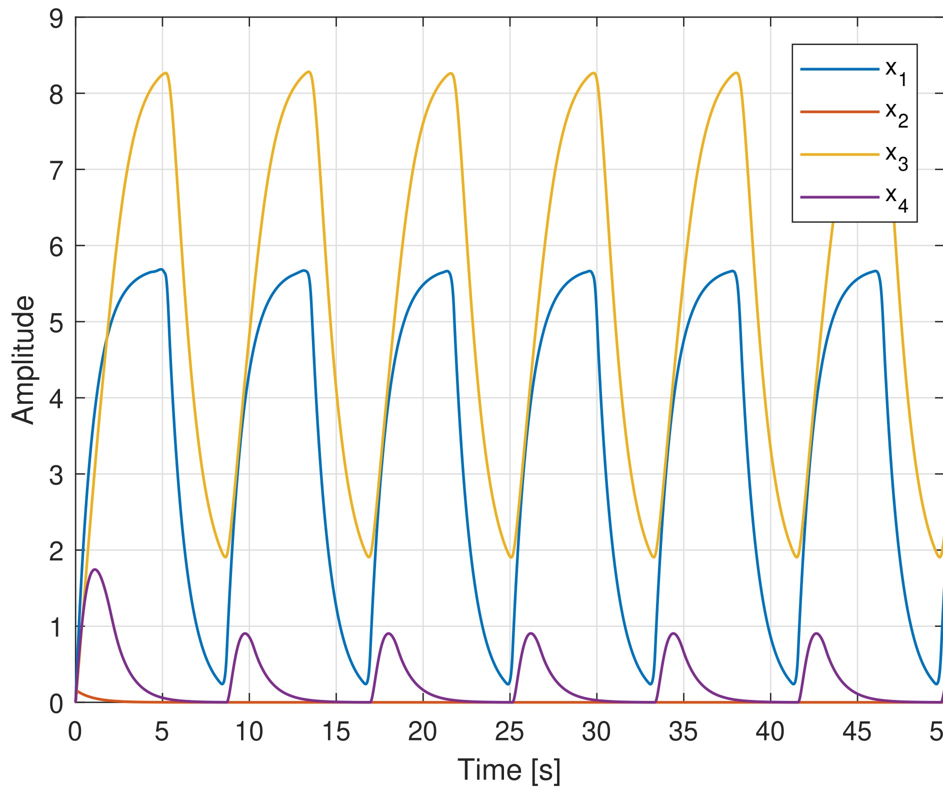


Figure 6.5: Temporal evolution of x_1, x_2, x_3, x_4, with x_1 being an excitatory node and form x_2 tp x_4 being inhibitory ones.

Corollary 6.3.7. ***(Sufficient conditions for the possible existence of non degenerate oscillations in a 1-excitatory - m-inhibiroty network with a semi closed form for u).*** *Consider a network with structure* $(\boldsymbol{W}, \boldsymbol{m})$ *with 1 excitatory node and m inhibitory nodes. Let* $\mathbb{L}$ *be the set of inhibitory nodes able to produce non-degenerate oscillations. For the existence of a* $\boldsymbol{u}$ *for which the system will not have any stable equilibria is sufficient for the network to verify:*

$$\begin{cases} \forall i \in \{2, \dots, m+1\} \text{ and } i \in \mathbb{L} & \\ \quad (1 - |W|_{1,1})m_1 + |W|_{1,i} m_i > 0 & (sufficient\ condition\ 1) \\ \quad -|W|_{i,1} m_1 + (|W|_{i,i} + 1) m_i < 0 & (sufficient\ condition\ 2) \\ \quad \forall j \neq i \text{ and } j \in \{2 \dots, m+1\} & \\ \qquad -|W|_{i,1} m_1 + |W|_{i,j} m_j > 0 & (sufficient\ condition\ 3) \\ \forall l \text{ set of nodes s.t } l \subseteq \mathbb{L} \text{ and } |l| \geq 2 & \\ \quad Let\ \boldsymbol{A} = \boldsymbol{W}(\boldsymbol{I} - \boldsymbol{\Sigma}^l \boldsymbol{W})^{-1} & \\ \quad \Big((diag(\boldsymbol{A}) + \big[\boldsymbol{A} - diag(\boldsymbol{A})\big]^{+})(\boldsymbol{I} - \Sigma^l)\boldsymbol{m} \Big)_1 < \boldsymbol{m}_1 & (sufficient\ condition\ 4) \end{cases} \tag{6.57}$$

With an input $\boldsymbol{u} \in \mathbb{U}$ *where* $\mathbb{U}$ *is:*

$$\mathbb{U} \subset \mathbb{R}^{m+1} = \prod_i \mathbb{U}_i \text{ where } \mathbb{U}_i = \begin{cases} (0, \varepsilon_i) & If\ i = 1 & (cond.\ 1) \\ (-\varepsilon_i, 0) & If\ i \in \{2, \dots, m+1\} \text{ and } i \in \mathbb{L} & \\ & \text{and } \varepsilon_i < |W|_{i,1} m_1 & (cond.\ 2) \\ (-\infty, -|W|_{i,1} m_1) & If\ i \in \{2, \dots, m+1\} \text{ and } i \notin \mathbb{L} & (cond.\ 3) \end{cases} \tag{6.58}$$

And ε_i *is:*

$$\begin{cases} \varepsilon_1 = \min\Big(\big(\min_{i \in \mathbb{L}} (1 - |W|_{1,1}) m_1 + |W|_{1,i} m_i \big), \big(\min_{\substack{\forall l \subseteq \mathbb{L} \\ |L| \geq 2}} \frac{m_1 - B^l}{\#[(A^l \Sigma^l)^-_{1,:}] + 1} \big) \Big) & \\ \varepsilon_i = \min\Big(\big(\min_{i \in \mathbb{L}} -W_{i,1} m_1 + (W_{i,i} + 1) m_i \big), \big(\min_{\substack{\forall l \subseteq \mathbb{L} \\ |L| \geq 2}} \frac{m_1 - B^l}{(\#[(A^l \Sigma^l)^-_{1,:}] + 1) A^l_{1,i}} \big) \Big) & \forall i \in \mathbb{L} \end{cases} \tag{6.59}$$

Example 6.3.8. *The following network is used to exemplify how this approach can lead to non-degenerate oscillations. The system is the following:*

$$\boldsymbol{W} = \begin{bmatrix} 14.7654 & -8.6380 & -8.0341 & -3.9274 & -9.4365 \\ 33.3003 & -4.1929 & -2.3322 & -4.2673 & -6.5820 \\ 16.1836 & -0.6326 & -3.4432 & -3.0320 & -3.8792 \\ 26.3386 & -4.3163 & -7.5901 & -0.8415 & -8.0899 \\ 26.2514 & -5.0468 & -7.1248 & -9.0519 & -0.7858 \end{bmatrix} \quad \boldsymbol{m} = \begin{bmatrix} 1.8134 \\ 7.3404 \\ 3.5702 \\ 7.2033 \\ 6.3083 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 0.6747 \\ -1.4978 \\ -0.2697 \\ -3.5327 \\ -0.6367 \end{bmatrix} \tag{6.60}$$

And the temporal evolution of the system is the following.

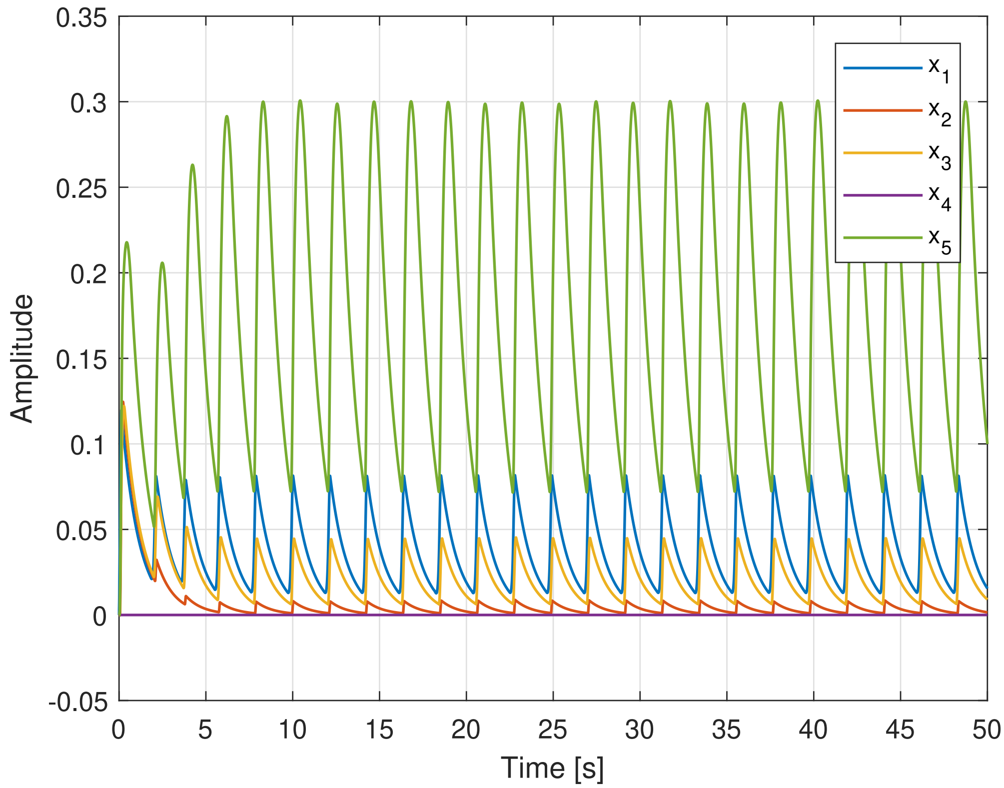


Figure 6.6: Temporal evolution of x_1, x_2, x_3, x_4, x_5, with x_1 being an excitatory node and form x_2 tp x_5 being inhibitory ones.

Chapter 7

Degenerate vs Non-degenerate Oscillations

7.1 Definitions and main results

Definition 7.1.1. ***(Degenerate Oscillations).***
A degenerate oscillation will be a non stable equilibrium temporal evolution of the dynamics (2.7)-(2.8) where only the minimum number of nodes for the existence of oscillations are not in a constant equilibrium state (i.e only the minimum number of node for the existence of oscillations are oscillating)

Lemma 7.1.2. ***(Manually choosing $\boldsymbol{u}$ to saturate the nodes).***
Let $\boldsymbol{u}$ be the input vector.
Let E be the set of excitatory nodes.
Let I be the set of inhibitory nodes.
If there $\exists k$ such that $u_k \notin [-\sum\limits_{e \in E} |W|_{k,e} m_e, \sum\limits_{i \in I} |W|_{k,i} m_i]$ then the node k will always temporally evolve towards negative or positive saturation.

Proof. In order to proof this, is first needed to remark two things:

- $x_0 \in X$ where $X = [0, m_1] \times \cdots \times [0, m_{m+1}]$
- If $x(t_0) \in X$ then $\forall t > t_0$ $x(t) \in X$.

The first remark is a condition of the system which is being studied.
The second remark will follow either from $\dot{x_k}(t) \geq 0$ if $x_k(t) = 0$ or from $\dot{x_k}(t) \leq 0$ if $x_k(t) = m_i$.
Both cases are analogous. If $x_k(t) = 0$ then $\tau_i \dot{x}_i(t) = -x_i(t) + [\boldsymbol{W}\boldsymbol{x}(t) + u_k]_0^{m_k} = 0 + [\boldsymbol{W}\boldsymbol{x}(t) + u_k]_0^{m_k} \geq 0$.
We will develop the proof for first case (second case).
Let's consider that $u_k < -\sum\limits_{e \in E} |W|_{k,e} m_e \big(> \sum\limits_{i \in I} |W|_{k,i} m_i \big)$ and check that $\lim_{t \to \infty} x_k(t) = 0$

($\lim_{t\to\infty} x_k(t) = m_k$).
For that we will consider the dynamics of the system:

$$\tau\dot{\boldsymbol{x}} = -\boldsymbol{x}(t) + [\boldsymbol{W}\boldsymbol{x}(t) + \boldsymbol{u}]_0^{\boldsymbol{m}} \tag{7.1}$$

That developing the terms:

$$\begin{bmatrix} \tau_{e_1}\dot{x}_{e_1} \\ \vdots \\ \tau_{e_n}\dot{x}_{e_n} \\ \tau_{i_1}\dot{x}_{i_1} \\ \vdots \\ \tau_{i_m}\dot{x}_{i_m} \end{bmatrix} = - \begin{bmatrix} x_{e_1} \\ \vdots \\ x_{e_n} \\ x_{i_1} \\ \vdots \\ x_{i_m} \end{bmatrix} + \left[\begin{bmatrix} |W|_{e_1,e_1} & \dots & |W|_{e_1,e_n} & -|W|_{e_1,i_1} & \dots & -|W|_{e_1,i_m} \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ |W|_{e_n,e_1} & \dots & |W|_{e_n,e_n} & -|W|_{e_n,i_1} & \dots & -|W|_{e_n,i_m} \\ |W|_{i_1,e_1} & \dots & |W|_{i_1,e_n} & -|W|_{i_1,i_1} & \dots & -|W|_{i_1,i_m} \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ |W|_{i_m,e_1} & \dots & |W|_{i_m,e_n} & -|W|_{i_m,i_1} & \dots & -|W|_{i_m,i_m} \end{bmatrix} \begin{bmatrix} x_{e_1} \\ \vdots \\ x_{e_n} \\ x_{i_1} \\ \vdots \\ x_{i_m} \end{bmatrix} + \begin{bmatrix} u_{e_1} \\ \vdots \\ u_{e_n} \\ u_{i_1} \\ \vdots \\ u_{i_m} \end{bmatrix} \right]_0^{\boldsymbol{m}} \leq$$

$$- \begin{bmatrix} x_{e_1} \\ \vdots \\ x_{e_n} \\ x_{i_1} \\ \vdots \\ x_{i_m} \end{bmatrix} + \left[\begin{bmatrix} |W|_{e_1,e_1} & \dots & |W|_{e_1,e_n} & -|W|_{e_1,i_1} & \dots & -|W|_{e_1,i_m} \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ |W|_{e_n,e_1} & \dots & |W|_{e_n,e_n} & -|W|_{e_n,i_1} & \dots & -|W|_{e_n,i_m} \\ |W|_{i_1,e_1} & \dots & |W|_{i_1,e_n} & -|W|_{i_1,i_1} & \dots & -|W|_{i_1,i_m} \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ |W|_{i_m,e_1} & \dots & |W|_{i_m,e_n} & -|W|_{i_m,i_1} & \dots & -|W|_{i_m,i_m} \end{bmatrix} \begin{bmatrix} m_{e_1} \\ \vdots \\ m_{e_n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} u_{e_1} \\ \vdots \\ u_{e_n} \\ u_{i_1} \\ \vdots \\ u_{i_m} \end{bmatrix} \right]_0^{\boldsymbol{m}} = \tag{7.2}$$

$$- \begin{bmatrix} x_{e_1} \\ \vdots \\ x_{e_n} \\ x_{i_1} \\ \vdots \\ x_{i_m} \end{bmatrix} + \begin{bmatrix} \sum\limits_{e\in E} |W|_{e_1,e} m_e + u_{e_1} \\ \vdots \\ \sum\limits_{e\in E} |W|_{e_n,e} m_e + u_{e_n} \\ \sum\limits_{e\in E} |W|_{i_1,e} m_e + u_{i_1} \\ \vdots \\ \sum\limits_{e\in E} |W|_{i_m,e} m_e + u_{i_m} \end{bmatrix}_0^{\boldsymbol{m}}$$

So for node k taking into account that $-\sum\limits_{e\in E} |W|_{k,e} m_e > u_k$ we will have:

$$\tau_i \dot{x}_k(t) \leq -x_k(t) + [\sum_{e\in E} |W|_{k,e} m_e + u_k]_0^{m_k} = -x_k(t)$$

And as $x_k(0) = (x_0)_k$ then we can ensure that $0 \leq x_k(t) \leq (x_0)_k e^{-t}$ so for the sandwich criteria $\lim_{t\to\infty} x_k(t) = 0$.

□

Remark 7.1.3. *The approaches of A and B sets (6.9, 6.10) of sufficient conditions (6.8) will manually saturate all the nodes except for the $j-k$ pair.*

7.2 Comparison on the degenearte rate of the different sets of sufficient conditions

The following section will compare the mean degenerate rate of both networks with an arbitrary structure and with the lack of stable equilibria. Before getting into the discussion let's first introduce some results:

Definition 7.2.1. ***(Degenerate Rate $\mathcal{D}$).***
For an specific system ($\boldsymbol{W}, \boldsymbol{m}, \boldsymbol{u}$) we will define the degenerate rate $\mathcal{D}$ as the quotient of the minimum nodes to generate oscillations and the actual nodes oscillating.
For nE-mI networks with $n \geq 1$ and $m \geq 1$ the analytic expression will be :

$$\mathcal{D} = \frac{2}{\{ \ \textit{Number of nodes oscillating} \ \}} \tag{7.3}$$

Note that if $\mathcal{D} = 2$ then there is a so called degenerate oscillation.
Also note that the degenerate rate is non dependent on the number of nodes in the network, so we will define the *Scaled non-degenerate rate* in order to give insight of how non-degenerate an oscillation is compared to the total number of nodes that compose the network.

Definition 7.2.2. ***(Scaled non-degenerate Rate).***
The scaled non-degenerate rate will be defined as:

$$sn\mathcal{D} = \frac{1 - \mathcal{D}}{N} \tag{7.4}$$

Where N is the number of nodes in the network.

7.2.1 Degeneration on arbitrary networks nE-mI

We will first aim to quantify the degenerate rate of all the existing oscillatory behavior in the brain. To do that, we will ignore our previous assumption that the oscillations steer from lack of stable equilibria and consider arbitrary oscillations, due to limit cycles or lack of stable equilibria, as well as arbitrary networks. In this line, multiple networks have been spanned using the following criteria:

$$\boldsymbol{W} = \begin{cases} W_{i,k} = \mathcal{U}(0, a_{max}) & If k \in \{1, \dots, n\} \text{ and } \forall i \\ W_{i,k} = \mathcal{U}(-d_{max}, 0) & If k \in \{1, \dots, n\} \text{ and } \forall i \end{cases} \tag{7.5}$$

$$\boldsymbol{m} = \begin{cases} m_i = \mathcal{U}(0, m_{max}) \quad \forall i \end{cases} \tag{7.6}$$

$$\boldsymbol{u} = \begin{cases} u_i = \mathcal{U}(-u_{min}, u_{max}) \quad \forall i \end{cases} \tag{7.7}$$

Where $a_{max}, d_{max}, m_{max} \in \mathbb{R}_{geq0}$ and u_{max} and u_{min} are choose to be:

$$\begin{cases} u_{max} = mean(|\boldsymbol{W}|)mean(\boldsymbol{m})m \\ u_{min} = mean(|\boldsymbol{W}|)mean(\boldsymbol{m})n \end{cases} \tag{7.8}$$

Networks up to 4E-4I have been considered and both the degenerate rate ($\mathcal{D}$) and the scaled non-degenerate rate($sn\mathcal{D}$) have been found to be:

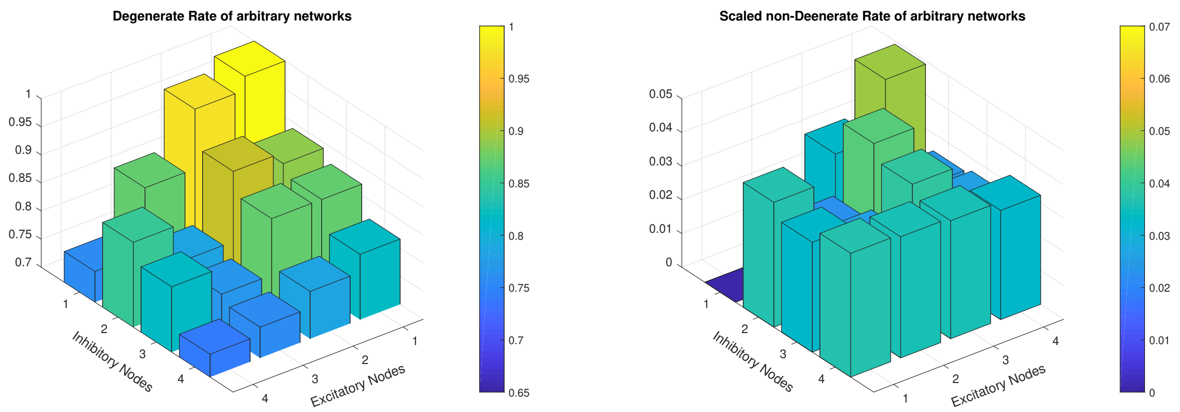


Figure 7.1: Degenerate and scaled non-degenerate rate for arbitrary networks up to 4E-4I.

It can be observed that the higher the dimension of the network, the lower the degenerate rate, as expected. More nodes will intuitively lead to more of them oscillating. However, the scaled non-degenerate reveals that the degenerate rate does not grow accordingly. This implies that the more nodes are found in the network the more it tends to become a winner takes it all evolution. Furthermore the degenerate rates are not lower than $\mathcal{D} = 0.7$ in any case, meaning that on average degenerate oscillations are still predominant predominate.

Remark 7.2.3. *As we can see, degenerate oscillations are predominant even in arbitrary networks. This direct implies that if we want to fully characterize the oscillatory behavior an understanding on the degenerate limit cycles is mandatory.*

7.2.2 Degenerate rate on arbitrary networks with lack of equilibria

On the previous chapter, multiple sets of sufficient conditions leading to networks with lack of stable equilibria have been proposed. The subsequent subsection will develop the results of degeneration regarding these sets.

Networks up to 8E-mI nodes have been considered for sets of conditions A,B, for both assumptions on lack of stability on the excitatory nodes, and networks up to 5E-4I have been considered for set C (the restrictive sufficient conditions are numerically difficult to verify when considering networks larger than those considered). Multiple networks have been spanned using the critera used before (7.5,7.6, 7.7) and its oscillatory behavior has been checked if the random generated network would satisfy the sufficient conditions.

The degenerate rate and the scaled non-degenerate for set A have been found to be:

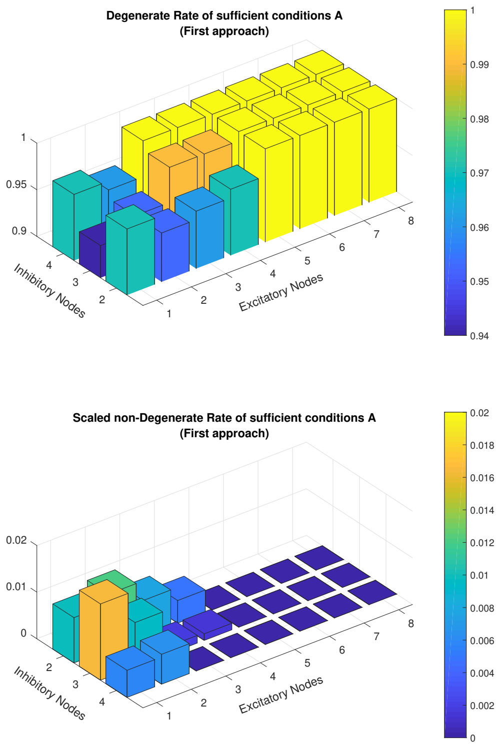
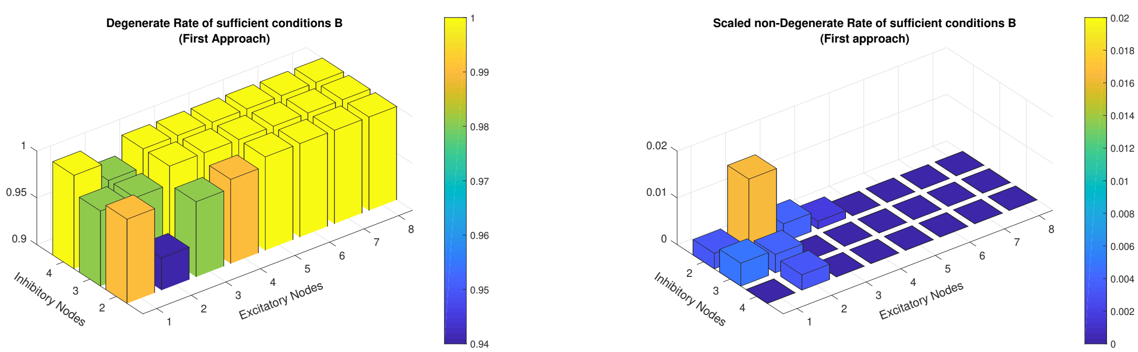


Figure 7.2: Degenerate Rate and scaled non-degenerate rate for networks satisfying set of sufficient conditions A, for boths approaches on the verification of the assumption (2), on networks up to 8E-4I

For set B, the results are



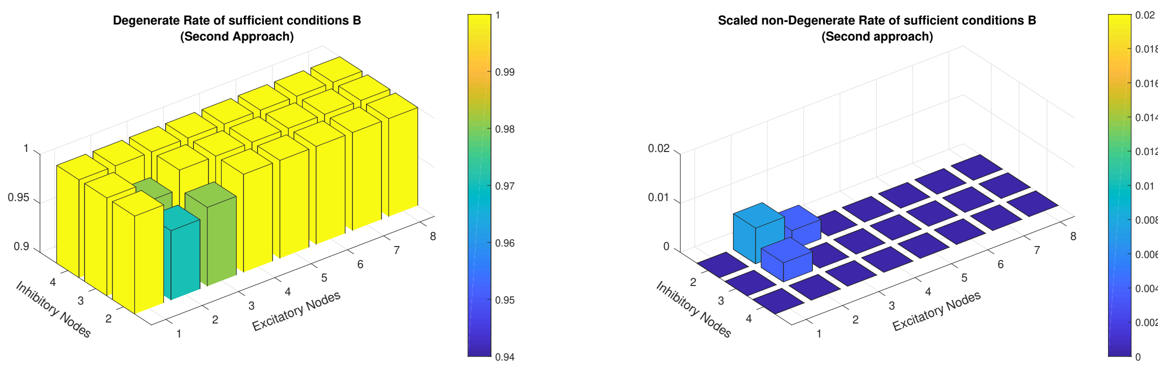


Figure 7.3: Degenerate Rate and scaled non-degenerate rate for networks satisfying set of sufficient conditions B, for boths approaches on the verification of the assumption (2), on networks up to 8E-4I

Finally for set C, we have:

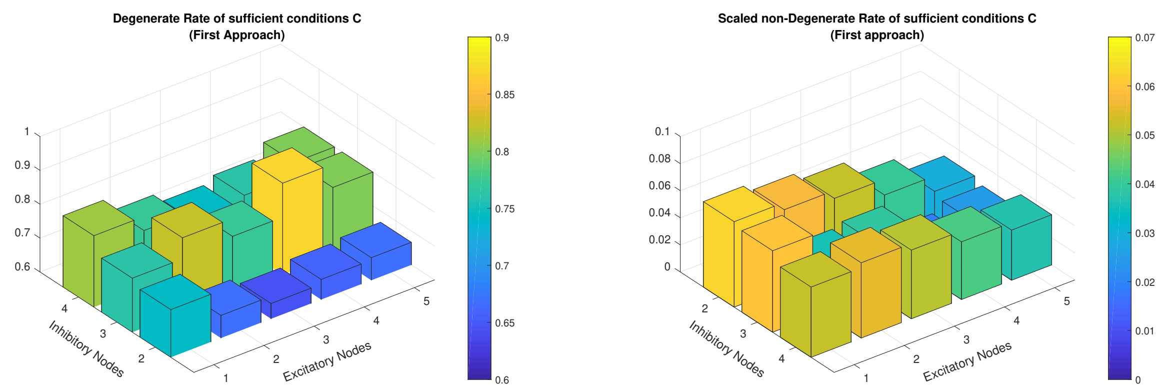


Figure 7.4: Rate and scaled non-degenerate rate for networks satisfying set of sufficient conditions C, for the simple approach on the verification of the assumption (2), on networks up to 5E-4I

On the networks with lack of stable equilibria, sets A and B, clearly lead to higher degenerate rates. It is an expected or comprehensible behavior as there are specific conditions imposed to the $j-k$ pair, the ones found to be oscillating. However the approach of both sets seems to emphasize a competition behavior over a cooperative one, which can be easily related to cognitive phenomena such as attention or focusing.

On the other hand, the sufficient conditions set C, even though no special condition has been imposed to any node, the oscillations on the network have higher degenerate rates than one would expect. This leads to lower scaled non-degenerate rates as the network grows. This oscillatory behavior could be easily related to more cooperative tasks where different regions compute and transmit information at the same time, leading to more complex oscillatory behavior.

Chapter 8

Pairwise unstable networks

8.1 Definitions and results

Definition 8.1.1. ***(Pairwise unstable Networks).***
A pairwise unstable network is a network structure $(\boldsymbol{W}, \boldsymbol{m})$ *such that for every switching region* $\boldsymbol{\Omega_{\sigma}}$*, involving only two different nodes in linear state, the system* $-\boldsymbol{I} + \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W}$ *will be unstable (i.e. the equilibrium candidate* $\boldsymbol{x}^*_{\boldsymbol{\sigma}}$ *will be unstable).*

Definition 8.1.2. ***(Fully Unstable Networks).***
A fully inhibitory network is a network structure $(\boldsymbol{W}, \boldsymbol{m})$ *such that for every switching region* $\boldsymbol{\Omega_{\sigma}}$*, involving two or more different nodes in linear state, the system* $-\boldsymbol{I} + \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W}$ *will be unstable (i.e. the equilibrium candidate* $\boldsymbol{x}^*_{\boldsymbol{\sigma}}$ *will be unstable).*

Remark 8.1.3. *Although either the pairwise instability and the fully instability of a network are properties defined for the whole structure of the network* $(\boldsymbol{W}, \boldsymbol{m})$*, it is an independent property from* $\boldsymbol{m}$*.*

Theorem 8.1.4. ***(Pairwise instability and fully unstable networks).***
Assume that the network structure $(\boldsymbol{W}, \boldsymbol{m})$ *verifies the assumptions of pairwise instability.*
If the determinant of every principal minor, $M_{j,i}$*, of the matrix* $-\boldsymbol{I}+\boldsymbol{W}$ *is negative* $(|M_{j,i}| < 0)$*, then for any switching region* $\boldsymbol{\Omega_{\sigma}}$ *involving more than two nodes in linear state the system* $-\boldsymbol{I} + \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W}$ *will be unstable, so the network will be fully unstable.*

Proof. Let's consider now a switching region $\boldsymbol{\Omega_{\sigma}}$ with an arbitrary set of arbitrary nodes K in linear state such that $|K| > 2$ (if $|K| = 2$ then $\boldsymbol{\Omega_{\sigma}}$ is unstable by hypothesis).
Following the construction of a matrix $\boldsymbol{P}^l$ like the one in the proof of (6.1.1).
The eigenvalues of the system $-\boldsymbol{I}+\boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W}$, will be (-1) of multiplicity $(N-|K|)$ and the eigenvalues of $\boldsymbol{P}^l$.

Considering the characteristic Polynomial of $\boldsymbol{P}^l$ we have:

$$Char(\boldsymbol{P}^l - \lambda \boldsymbol{I}) = (-1)^{|N|}\lambda^{|N|} + (-1)^{|N|-1}K_{|N|-1}\lambda^{|N|-1} + \cdots + (-1)K_1\lambda + det(\boldsymbol{P}^l) \tag{8.1}$$

Where $K_k = \sum_{\forall j} C_{k,j}$ where $C_{k,j}$ are all the determinants of the principal minors of size $|N| - k$. Looking into the term $K_{|N|-2}$ we have that:

$$K_{|N|-2} = \sum_{\forall i,j \in K} |M_{i,j}| \tag{8.2}$$

Where $M_{i,j}$ is the minor i, j of the matrix $-\boldsymbol{I} + \boldsymbol{W}$.
So, using the hypothesis, $K_{|N|-2} < 0$ and then it will always hold that $sgn((-1)^{|N|}) \neq sgn((-1)^{|N|-1}K_{|N|-2})$.
Using Routh criteria this implies that it always $\exists \lambda_i$ such that $Re(\lambda_i) > 0$. □

Lemma 8.1.5. ***(Sufficient conditions on the coefficients for a Fully Unstable Network).*** *If a network* $(\boldsymbol{W}, \boldsymbol{m})$ *verifies that*

$$\begin{cases} (d_{i,i}+1) < d_{j,i} & \forall j \in \{1, \dots m\} \\ (d_{i,i}+1) > b_{j,i} & \forall j \in \{1, \dots m\} \\ (a_{i,i}-1) > c_{j,i} & \forall j \in \{1, \dots n\} \\ (a_{i,i}-1) < a_{j,i} & \forall j \in \{1, \dots n\} \end{cases} \tag{8.3}$$

and that

$$a_{i,i} > d_{k,k} + 2 \qquad \forall i \in \{1, \dots, n\} \text{ and } \forall k \in \{1, \dots, m\} \tag{8.4}$$

Then the network verifying the assumption (2) on the instability of the excitatory nodes, and it will be a fully unstable network.

Proof. That the network structure satisfies assumption (2) is a direct consequence of theorem (6.1.1) because of it being a fully unstable and verifying that $a_{i,i} > d_{k,k} + 2 \ \forall i, k$.
So in order to proof that the network is fully unstable we will proof that it is, indeed, pairwise unstable and that it also verifies that $|M_{j,i}| < 0 \ \forall j, i$.
In order to proof pairwise instability let's go through all the possible cases of i, j: E-E, E-I, I-I.
For each one, if we consider $\boldsymbol{\Omega_\sigma}$ to be the corresponding switching region where only i and j are in linear state then the eigenvalues of full system $-\boldsymbol{I} + \boldsymbol{\Sigma}^l\boldsymbol{W}$ will be (-1) with multiplicity $(N-2)$ and the eigenvalues of the sub-matrices $(E-E)_{i,j}$, $(E-I)_{i,j}$, $(I-I)_{i,j}$. So let's check that each submatrix will always have an eigenvalue λ_i for which $Re(\lambda_i) > 0$.

- **E-E**
  The sub-matrix $(E-E)_{i,j}$ will be:

$$(E-E)_{i,j} = \begin{bmatrix} a_{i,i}-1 & a_{i,j} \\ a_{j,i} & a_{j,j}-1 \end{bmatrix} \tag{8.5}$$

As $a_{i,i} > 1$, then $Tr((E-E)_{i,j}) > 0$, so at least there exists one eigenvalue λ_i for which $Re(\lambda_i) > 0$, meaning that the system will be unstable for this specific switching region.

- **E-I**
  The sub-matrix $(E-I)_{i,j}$ will be:

$$(E-I)_{i,j} = \begin{bmatrix} a_{i,i}-1 & -b_{i,j} \\ c_{j,i} & -d_{j,j}-1 \end{bmatrix} \tag{8.6}$$

  Again, as $a_{i,i} > d_{j,j} + 2 \ \forall j$, then $Tr((E-I)_{i,j}) > 0$, so at least there exists one eigenvalue λ_i for which $Re(\lambda_i) > 0$.

- **I-I**
  The sub-matrix $(I-I)_{i,j}$ will be:

$$(I-I)_{i,j} = \begin{bmatrix} -d_{i,i}-1 & -d_{i,j} \\ -d_{j,i} & -d_{j,j}-1 \end{bmatrix} \tag{8.7}$$

  As $Tr((I-I)_{i,j}) < 0$, at least there exists one eigenvalue λ_i for which $Re(\lambda_i) < 0$. By inspection we can affirm that $Det(I-I_{i,j}) = \lambda_i\lambda_j = (d_{i,i}+1)(d_{j,j}+1) - d_{i,j}d_{i,j} < 0$.
  So, if both eigenvalues are real then it is trivial that $\exists\lambda_i > 0$, in particular $Re(\lambda_i) > 0$, so the system will be unstable.
  Now let's see that the eigenvalues λ_i and λ_j cannot be imaginary. Considering the characteristic polynomial we have that:

$$Char((I-I)_{i,j} - \lambda\boldsymbol{I}) = \lambda^2 + (d_{i,i}+1+d_{i,i}+1)\lambda + ((d_{i,i}+1)(d_{j,j}+1) - d_{i,j}d_{j,i} = 0 \tag{8.8}$$

  So the eigenvalues:

$$\lambda_k = \frac{-(d_{i,i}+1+d_{i,i}+1) \pm \sqrt{(d_{i,i}+1+d_{i,i}+1)^2 - 4((d_{i,i}+1)(d_{j,j}+1) - d_{i,j}d_{j,i})}}{2} \tag{8.9}$$

  However, as $-4((d_{i,i}+1)(d_{j,j}+1) - d_{i,j}d_{j,i}) > 0$ both eigenvalues must be real.

Now, it only remains to check that $|M_{i,j}| < 0 \forall i,j$.
For the case $(I-I)_{i,j}$ we already check by inspection. Regarding $(E-E)_{i,j}$ we trivially have that $|M_{i,j}| = (a_{i,i}-1)(a_{j,j}-1) - a_{i,j}a_{j,i} < 0$ and regarding $(E-I)_{i,j}$ we have that $|M_{i,j}| = -(a_{i,i}-1)(d_{j,j}+1) + b_{i,j}c_{j,i} < 0$.
So by theorem (8.1.4) we can affirm that the network will be fully unstable.

□

Chapter 9

Fully Inhibitory Networks

9.1 Motivation

A gamma wave is a pattern of neural oscillation in humans with a frequency between 25 and 100 Hz,[7] though 40 Hz is typical. They are though to be implicated in modeling and creating the conscious perception of the individuals.
Modern studies have shown that gamma oscillations are present and are clearly correlated with mostly inhibitory networks, so to give insight on how they can occur we will characterize the fully inhibitory networks.

9.2 Fully inhibitory networks

Definition 9.2.1. *Let network structure* $(\boldsymbol{W}, \boldsymbol{m})$ *following the dynamics (2.7). We will say that a network is fully inhibitory if and only if only inhibitory nodes are present in the network, i.e. the network structure* $\boldsymbol{W}$ *will be such that:*

$$\boldsymbol{W} = \begin{bmatrix} -d_{1,1} & -d_{1,2} & \dots & -d_{1,m} \\ -d_{2,1} & -d_{2,2} & \dots & -d_{2,m} \\ \vdots & \ddots & \dots & \vdots \\ -d_{m,1} & -d_{m,2} & \dots & -d_{m,m} \end{bmatrix} \tag{9.1}$$

Theorem 9.2.2. ***(Necessary condition on $\boldsymbol{W}$ for oscillatory behavior in Fully inhibitory networks).***
It is necessary for the network to admit oscillatory behavior that:

$$\boldsymbol{I} - \boldsymbol{W} \notin P \tag{9.2}$$

Where P is the set of P-matrices.

Proof. We will first proof that it is always necessary for $\boldsymbol{x}^* \in \boldsymbol{\Omega_\sigma}$ for a specific $\boldsymbol{\sigma}$.

Let $f(x) = [\boldsymbol{Wx} + \boldsymbol{u}]_0^{\boldsymbol{m}}$, describing the equilibrium points of the system.
f is clearly and application from $[0, m_1] \times \cdots \times [0, m_m]$ to itself, and as a switched affine system it is continuous.

Using Brouwer's fixed-point theorem, it always exists a fixed point in $[0, m_1] \times \cdots \times [0, m_m]$. This fixed point will be an equilibrium point of the original system and by construction it will always belong to a switching region, as $[0, m_1] \times \cdots \times [0, m_m] = \bigcup_{\boldsymbol{\sigma}} \boldsymbol{\Omega_\sigma}$
However, $\forall \boldsymbol{\Omega_\sigma}$ and all possible equilibrium candidates, these equilibria will always be stable so dynamics will not oscillate.
Let's proof it:
Consider and arbitrary region $\boldsymbol{\Omega_\sigma}$. The dynamics for the system in this switching region will steer from the matrix:

$$-\boldsymbol{I} + \boldsymbol{\Sigma}^l \boldsymbol{W} \tag{9.3}$$

Where l is the set of nodes in linear state.
Let $\boldsymbol{\Pi_\sigma}$ be such that $\boldsymbol{\Pi_\sigma \sigma} = [\boldsymbol{\Pi_\sigma} \sigma_{0,s}, \underbrace{l, \ldots, l}]$ the permutation matrix that matches the nodes into linear state to the right bottom of the matrix. Then we would have:

$$\boldsymbol{\Pi_\sigma}(-\boldsymbol{I} + \boldsymbol{\Sigma}^l \boldsymbol{W})\boldsymbol{\Pi_\sigma}^T = \begin{bmatrix} \boldsymbol{I} & 0 \\ * & \boldsymbol{P} \end{bmatrix} \tag{9.4}$$

The eigenvalues of the system, which will determine its instability or stability, will be (-1) of multiplicity $(m - l)$ and the eigenvalue of a matrix P where P will always be a principal minor of the original matrix. Note that the minor is also a $P-$matrix.
Let's now consider $(-P)$, that will be a non-negative matrix.
Using theorem 1 from [8], a sign symmetric P-matrix is stable, meaning for stability of P-matrices that all the eigenvalues fall at the positive complex quadrant, so all the eigenvalues of $-P$ will fall into the positive complex quadrant. Then the eigenvalues of P will fall in the negative complex quadrant and the system will be stable for any $\boldsymbol{\Omega_\sigma}$.
It is also a direct consequence of the Sylvester criterion [9].

□

Theorem 9.2.3. ***(On the stability of 2 node fully inhibitory networks).***
*Let a network structure be defined by $(\boldsymbol{W}, \boldsymbol{m})$, the dynamics (**??**) and composed only by two inhibitory nodes. The structure of $\boldsymbol{W}$ and $\boldsymbol{m}$ will be:*

$$\boldsymbol{W} = \begin{bmatrix} -d_{1,1} & -d_{1,2} \\ -d_{2,1} & -d_{2,2} \end{bmatrix} \qquad \boldsymbol{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \tag{9.5}$$

Then $\forall \boldsymbol{u}$ input of the system, the system will be stable.

Proof. We will proof that for any $\boldsymbol{u}$ there always $\exists$ at least one $\boldsymbol{\Omega_\sigma}$ stable switching region containing its equilibrium point.
We will distinguish two different cases:

- $(d_{1,1}+1)(d_{2,2}+1) - d_{1,2}d_{2,1} > 0$
  If this happens then $\boldsymbol{I} - \boldsymbol{W} \in P$, so the necessary condition defined by theorem (9.2.2) will not be fulfilled.

- $(d_{1,1}+1)(d_{2,2}+1) - d_{1,2}d_{2,1} < 0$
  The only unstable region is the one where both nodes are found in linear state. The invalid inputs, where the equilibrium point will fall inside its corresponding region, are by theorem (5.1.1) the inputs $\boldsymbol{u}$ such that :

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} (d_{1,1}+1)m_1 \\ d_{2,1}m_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} d_{1,2}m_2 \\ (d_{2,2}+1)m_2 \end{bmatrix} \tag{9.6}$$

  Where α_1,$\alpha_2 \in [0,1]$.
  We want to proof that for every $\boldsymbol{u}$ belonging to this parallelogram, then the equilibrium point of some of the regions where only one node is in linear will belong to its corresponding switching region, providing an stable equilibrium for the system.

The six possible sets of inequalities describing the $\boldsymbol{u}$ for which the equilibrium point will lay inside the region are:

$$\begin{aligned}
&\boldsymbol{\sigma} = (l,0) = \begin{cases} u_1 \in [0,(d_{1,1}+1)m_1] & \textit{and} \\ u_2 < \frac{d_{2,1}}{d_{1,1}+1}u_1 \end{cases} && \boldsymbol{\sigma} = (s,l) = \begin{cases} u_1 < \frac{d_{1,2}}{d_{2,2}+1}(u_2 - d_{2,1}m_1) + (d_{1,1}+1)m_1 & \textit{and} \\ u_2 \in [d_{2,1}m_1, (d_{2,2}+1)m_2 + d_{2,1}m_1] \end{cases} \\
&\boldsymbol{\sigma} = (0,l) = \begin{cases} u_1 < \frac{d_{1,2}}{d_{2,2}+1}u_2 & \textit{and} \\ u_2 \in [0,(d_{2,2}+1)m_2] \end{cases} && \boldsymbol{\sigma} = (l,s) = \begin{cases} u_1 \in [d_{1,2}m_2, (d_{1,1}+1)m_1 + d_{1,2}m_2] & \textit{and} \\ u_2 < \frac{d_{2,1}}{d_{2,2}+1}(u_1 - d_{1,2}m_2) + (d_{2,2}+1)m_2 \end{cases} \\
&\boldsymbol{\sigma} = (s,0) = \begin{cases} u_1 > (d_{1,1}+1)m_1 & \textit{and} \\ u_2 < d_{2,1}m_1 \end{cases} && \boldsymbol{\sigma} = (0,s) = \begin{cases} u_1 > d_{1,1}m_1 & \textit{and} \\ u_2 > (d_{2,2}+1)m_2 \end{cases}
\end{aligned} \tag{9.7}$$

Suppose that $u_1 \in [0,(d_{1,1}+1)m_1]$, then we want to check that $u_2 < \frac{d_{2,1}}{d_{1,1}+1}u_1$. Suppose that $\boldsymbol{u} \in Q_{1,2}$ then, by substituting we have:

$$\begin{aligned}
\frac{d_{2,1}}{d_{1,1}+1}(\alpha_1(d_{1,1}+1)m_1 + \alpha_2 d_{1,2}m_2) &> \alpha_1 d_{2,1}m_1 + \alpha_2(d_{2,2}+1)m_2 \\
\alpha_1 d_{2,1}m_1 + \frac{d_{2,1}d_{1,2}}{d_{1,1}+1}\alpha_2 &> \alpha_1 d_{2,1}m_1 + \alpha_2(d_{2,2}+1)m_2 \\
\frac{d_{2,1}d_{1,2}}{d_{1,1}+1} &> (d_{2,2}+1)
\end{aligned} \tag{9.8}$$

As this is true by hypothesis if $\boldsymbol{u} \in Q_{1,2}$ and $u_1 \in [0,(d_{1,1}+1)m_1]$ then $\boldsymbol{u} \in (l,0)$ and the system will be stable.

Suppose that $u_2 \in [0, (d_{2,2}+1)m_2]$, then we want to check that $u_1 < \frac{d_{1,2}}{d_{2,2}+1} u_2$. Suppose that $\boldsymbol{u} \in Q_{1,2}$ then, by substituting we have:

$$\begin{aligned}\frac{d_{1,2}}{d_{2,2}+1}(\alpha_1 d_{2,1} m_1 + \alpha_2 (d_{2,2}+1) m_2) > (\alpha_1 (d_{1,1}+1) m_1 + \alpha_2 d_{1,2} m_2)& \\ \alpha_1 m_1 \frac{d_{1,2} d_{2,1}}{d_{2,2}+1} + \alpha_2 d_{1,2} m_2 > \alpha_1 (d_{1,1}+1) m_1 + \alpha_2 d_{1,2} m_2& \\ \frac{d_{1,2} d_{2,1}}{d_{2,2}+1} > (d_{1,1}+1)&\end{aligned} \tag{9.9}$$

As this is true by hypothesis, if $\boldsymbol{u} \in Q_{1,2}$ and $u_2 \in [0, (d_{2,2}+1)m_2]$ then $\boldsymbol{u} \in (0, l)$ and the system will be stable.
Now let f be:

$$\begin{aligned} f \colon \mathbb{U} &\longrightarrow \mathbb{U} \\ \boldsymbol{u} &\longmapsto \boldsymbol{f}(\boldsymbol{u}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 - (d_{1,1}+1)m_1 - d_{1,2} m_2 \\ u_2 - d_{2,1} m_1 - (d_{2,2}+1) m_2 \end{bmatrix} \end{aligned} \tag{9.10}$$

Note that if $\boldsymbol{u} \in Q_{1,2}$ then $f(\boldsymbol{u}) \in Q_{1,2}$, because considering a $\boldsymbol{u} \in Q_{1,2}$ then we have:

$$\begin{aligned} f\Big(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\Big) &= \begin{bmatrix} -(\alpha_1 (d_{1,1}+1) m_1 + \alpha_2 d_{1,2} m_2 - (d_{1,1}+1) m_1 - d_{1,2} m_2) \\ -(\alpha_1 d_{2,1} m_1 + \alpha_2 (d_{2,2}+1) m_2 - d_{2,1} m_1 - (d_{2,2}+1) m_2) \end{bmatrix} \\ &= \begin{bmatrix} (1-\alpha_1)(d_{1,1}+1) m_1 + (1-\alpha_2) d_{1,2} m_2 \\ (1-\alpha_1) d_{2,1} m_1 + (1-\alpha_2)(d_{2,2}+1) m_2 \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 (d_{1,1}+1) m_1 + \beta_2 d_{1,2} m_2 \\ \beta_1 d_{2,1} m_1 + \beta_2 (d_{2,2}+1) m_2 \end{bmatrix} \end{aligned} \tag{9.11}$$

With $\beta_1, \beta_2 \in [0, 1]$.
Note that the image through f of the $\boldsymbol{u}$ verifying the inequalities gives the the image verifies equivalent inequalities to the ones in negative saturation, (l,s) and (s,l) are equivalent through f to (l,0) and (0,l), respectively. Let's see it with (s,l), then (l,s) will be analogue.
Abusing notation:

$$f(\boldsymbol{u} \in (s, l)) =$$
$$= \begin{cases} -u_1 + (d_{1,1}+1) m_1 + d_{1,2} m_2 > \frac{d_{1,2}}{d_{2,2}+1}((d_{2,2}+1) m_2 + d_{2,1} m_1 - u_2 - d_{2,1} m_1) + (d_{1,1}+1) m_1 \\ d_{2,1} m_1 < -u_2 + d_{2,2}+1) m_2 + d_{2,1} m_1 < +d_{2,2}+1) m_2 + d_{2,1} m_1 \end{cases}$$

$$
\begin{aligned}
&= \begin{cases} -u_1 + d_{1,2}m_2 > d_{1,2}m_2 - \frac{d_{1,2}}{d_{2,2}+1}u_2 \\ -(d_{2,2}+1)m_2 < -u_2 < 0 \end{cases} \\
&= \begin{cases} u_1 < \frac{d_{1,2}}{d_{2,2}+1}u_2 \\ 0 < u_2 < (d_{2,2}+1)m_2 \end{cases}
\end{aligned} \tag{9.12}
$$

As f is bijection, then $\boldsymbol{u} \in (s, l)$ $\big(\boldsymbol{u} \in (l, s)\big)$ if and only if $f(\boldsymbol{u}) \in f(\boldsymbol{u} \in (s, l)) = (0, l)$ $\big(f(\boldsymbol{u}) \in f((l, s)) = (l, 0)\big)$.
So if $\boldsymbol{u} \in Q_{1,2}$ and $u_1 \in [d_{1,2}m_2, d_{1,2}m_2 + (d_{1,1}+1)m_1]$ then $f(\boldsymbol{u})_1 = [0, (d_{1,1}+1)m_1]$ and as $f(\boldsymbol{u}) \in Q_{1,2}$ then $f(\boldsymbol{u}) \in (l, 0)$ implying that $\boldsymbol{u} \in (l, s)$ and the system will have an stable equilibrium point.
On the other hand, if $\boldsymbol{u} \in Q_{1,2}$ and $u_2 \in [d_{2,1}m_1, d_{2,1}m_1 + (d_{2,2}+1)m_2]$ then $f(\boldsymbol{u})_2 = [0, (d_{2,2}+1)m_2]$ and as $f(\boldsymbol{u}) \in Q_{1,2}$ then $f(\boldsymbol{u}) \in (0, l)$, as seen before, implying that $\boldsymbol{u} \in (s, l)$ and the system will have an stable equilibrium point.
Finally if $\boldsymbol{u} \in Q_{1,2}$ and

$$
\begin{cases} u_1 \in [(d_{1,1}+1)m_1, d_{2,1}m_1] \quad \textit{and} \\ u_2 \in [(d_{2,2}+1)m_2, d_{1,2}m_2] \end{cases} \tag{9.13}
$$

Then either $\boldsymbol{u} \in \sigma = (s, 0)$ or $\boldsymbol{u} \in \sigma = (s, 0)$.

The analytic proof might not give fully insight or understanding of the behavior of the inequalities if $(d_{1,1}+1)(d_{2,2}+1) - d_{1,2}d_{2,1} < 0$. To clarify the explanation, the following figure represents the different invalid zones for this specific case, bringing the intuition into the matter:

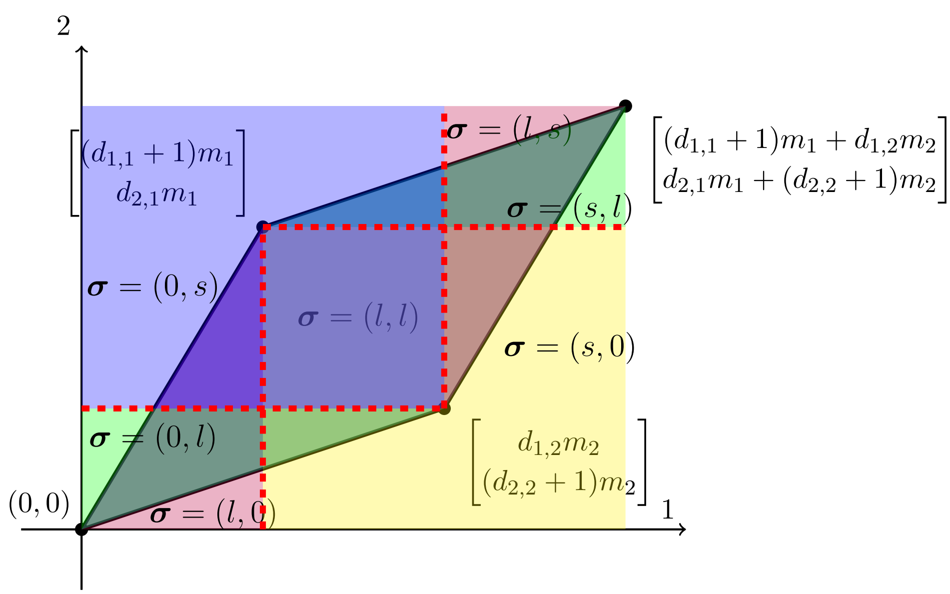


Figure 9.1: Graphical representation of the invalid regions for $\boldsymbol{u}$.

□

9.3 Pairwise Unstable Fully Inhibitory networks

Definition 9.3.1. *A pairwise unstable fully inhibitory network is a network structure* $(\boldsymbol{W}, \boldsymbol{m})$ *with only inhibitory nodes present in the network and such that for every switching region* $\boldsymbol{\Omega_\sigma}$*, involving only two different nodes in linear state, the system* $-\boldsymbol{I} + \boldsymbol{\Sigma}^l_{\boldsymbol{\sigma}}\boldsymbol{W}$ *will be unstable.*

Remark 9.3.2. *As the network is pairwise unstable the determinant each principal minor involving two nodes* $M_{i,j}$ *will be negative, i.e.* $|M_{i,j}| < 0$.

Remark 9.3.3. *For theorem (8.1.4) the network will be fully unstable.*

Remark 9.3.4. *In order to oscillate, for theorem (9.2.3), the network will be composed by at least three nodes.*

Remark 9.3.5. *As the network is fully unstable, the only switching regions where the corresponding equilibrium candidates will be stable, and cannot be contained inside the corresponding region, are the ones with only one node in linear or the ones with non node in linear state.*

Lemma 9.3.6. ***(Linear inequalities for sufficient small*** $\boldsymbol{u}$***).***
Let C *be the region bounded by the one node in saturation switching regions:*

$$C = \left[0, \min_i(\boldsymbol{m}(\boldsymbol{I}-\boldsymbol{W})_{1,:})_i\right) \times \left[0, \min_i(\boldsymbol{m}(\boldsymbol{I}-\boldsymbol{W})_{2,:})_i\right) \times \cdots \times \left[\min_i(\boldsymbol{m}(\boldsymbol{I}-\boldsymbol{W})_{m,:})_i\right) \tag{9.14}$$

Then, if $\boldsymbol{u} \in C$*, it is necessary and sufficient for the system not to have any stable equilibria that* $\boldsymbol{u} \in \bigcap_{i=0}^m T_i^C$ *where:*

$$T_i^C = \begin{cases} \boldsymbol{u} \mid \;\; \exists u_j > 0 & \textit{If } i = 0 \\ \boldsymbol{u} \mid \;\; x^*_{\boldsymbol{\sigma}_i} \notin \boldsymbol{\Omega}_{\boldsymbol{\sigma}_i} \textit{ where } \sigma_i \textit{ has the ith node in linear} & \\ \qquad\qquad \textit{state and the other ones in negative saturation} & \textit{If } i \in \{1, \dots, m\} \end{cases} \tag{9.15}$$

The inequalities describing T_i^C *for* $i \in \{1, \dots, m\}$*:*

$$T_i^C = \bigcup_{j \neq i} (u_j > \frac{|W|_{j,i}}{|W|_{i,i}+1} u_i) \quad \forall i \in \{1, \dots, m\} \tag{9.16}$$

Proof. If $\boldsymbol{u} \in C$ then trivially no switching region where a node is found in positive saturation can contain its equilibrium point. Suppose that node i is in positive saturation, then the evolution of this node will be:

$$\dot{x_i}(t) = -x_i + [-\sum_{t \neq i} W_{i,t}x_t - W_{i,i}m_i + u_i]_0^{m_i} \tag{9.17}$$

As $u_i < \min_j(\boldsymbol{m}(\boldsymbol{I}-\boldsymbol{W})_{i,:})_j$, in particular $u_i < W_{i,i}m_i$ the node will always go out of positive saturation. The other bounds are to keep $\boldsymbol{u}$ always in the one node in linear switching regions.
Then, the set of inequalities T_i^C represent only the ones involving on node in linear and no other

node in positive saturation. For all those regions $\boldsymbol{\sigma}$, $\boldsymbol{u} \in \bigcap_{i=0}^{m} T_i^C$ will provide that $\boldsymbol{x}^*_{\boldsymbol{\sigma}} \notin \boldsymbol{\Omega}_{\boldsymbol{\sigma}}$. □

Remark 9.3.7. *Consider two arbitrary nodes i, j.*
If $u_j > \frac{|W|_{j,i}}{|W|_{i,i}+1} u_i$ then $u_i < \frac{|W|_{i,j}}{|W|_{j,j}+1} u_j$, and viceversa (i.e. Both inequalities cannot be held at the same time).

Proof. Suppose $u_j > \frac{|W|_{j,i}}{|W|_{i,i}+1} u_i$) then $u_i < \frac{|W|_{i,i}+1}{|W|_{j,i}} u_j < \frac{|W|_{i,j}}{|W|_{j,j}+1} u_j$ as $(|W|_{i,i}+1)(|W|_{j,j}+1) - |W|_{i,j}|W|_{j,i} < 0$. □

Definition 9.3.8. ***(Notation).***
Let F be the following adjacency matrix:

$$F = \begin{bmatrix} 0 & \frac{|W|_{1,1}+1}{|W|_{2,1}} & \frac{|W|_{1,1}+1}{|W|_{3,1}} & \cdots & \frac{|W|_{1,1}+1}{|W|_{m,1}} \\ \frac{|W|_{2,2}+1}{|W|_{1,2}} & 0 & \frac{|W|_{2,2}+1}{|W|_{3,2}} & \cdots & \frac{|W|_{2,2}+1}{|W|_{m,2}} \\ \frac{|W|_{3,3}+1}{|W|_{1,3}} & \frac{|W|_{3,3}+1}{|W|_{2,3}} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{|W|_{m,m}+1}{|W|_{1,m}} & \frac{|W|_{m,m}+1}{|W|_{2,m}} & \frac{|W|_{3,3}+1}{|W|_{m,3}} & \cdots & 0 \end{bmatrix} \tag{9.18}$$

Let $G = (V, E)$ be the corresponding graph.
Let $G_c = (V_c, E_c)$ be the sub-graph of G, $E_c \subseteq E$ and $V_c \subseteq V$, such that for each node there is one and only one outaking edge and if $u - v \in E_c$ then $v - u \notin E_c$.
Let F_c be the adjacency matrix corresponding to G_c.

Remark 9.3.9. *Note that if we are considering a pairwise unstable fully inhibitory network then, as $(|W|_{i,i}+1)(|W|_{j,j}+1) - |W|_{i,j}|W|_{j,i} < 0$ then necessary $|W|_i, j > 0$ $\forall i, j$.*

Remark 9.3.10. *A cycle will always exist in G_c.*

Proof. Suppose it does not. Then pick one arbitrary node v and follow the path to $v - u$, that will always exist by assumption. As the number of nodes is finite if there is not a cycle at least one node does not have any out-coming edge. Contradiction.
Also the length of the cycle will be at least 3 nodes. □

Remark 9.3.11. *The single non-zero row value of F_c can be interpreted as the inequality of T_i we choose to verify. That's why at least have to be one non single value in each row. We can assume that there is only one because the case were there are more than one (i.e more than one inequality is satisfied) it can be divided into two sub-cases.*

Definition 9.3.12. ***(Valid cycle).***
Let h be a cycle in some G_c.
h will be a valid cycle if the adjacency matrix of the cycle A has an $\rho(A) > 1$.

Theorem 9.3.13. *It is sufficient for the pairwise unstable fully inhibitory network $(\boldsymbol{W}, \boldsymbol{m})$ to not to have any stable equilibira that $\exists$ at least one valid cycle.*

Proof. Let A be the adjacency matrix of the cycle. Note that A is a matrix of, at least dimension 3, that will, for both the columns and the rows, have only one non zero element.
As A is the adjacency matrix of a cycle then A is globally reachable, equivalent to being irreducible. For the Perron–Frobenius theorem [10] for irreducible matrices we can affirm the following things about the eigenvalues and the eigenvectors of A:

- The spectral radius of A, $r = \rho(A)$, is a positive real number and it is an eigenvalue of the matrix A, called the Perron–Frobenius eigenvalue.
- A has a right eigenvector $\boldsymbol{v}$ with eigenvalue r whose components are all positive.
- The only eigenvectors whose components are all positive are those associated with the eigenvalue r.

Then let $\boldsymbol{v}$ be the eigenvector of the perron-Forbenius eigenvalue. We can affirm that:

$$A\boldsymbol{v} = r\boldsymbol{v} > \boldsymbol{v} \tag{9.19}$$

So if we choose a $\lambda \in (0, 1)$ and for every node i in the cycle we set u_i to be:

$$u_i = \frac{v_i}{\|\boldsymbol{v}\|} \lambda \min_{i,j}((\boldsymbol{I} - \boldsymbol{W})_{i,j} \tag{9.20}$$

And for the j not in the cycle we set $u_j = 0$, we will have constructed an input $\boldsymbol{u}$ that will verify all the inequalities.

□

Corollary 9.3.14. *The valid cycle condition can be seen as the existence of a chain of nodes* $i_1, \ldots, \imath_m$ *such that :*

$$\frac{|W|_{i_1,i_1} + 1}{|W|_{i_2,i_1}} \cdot \frac{|W|_{i_2,i_2} + 1}{|W|_{i_3,i_2}} \cdot \ldots \cdot \frac{|W|_{i_m,i_m} + 1}{|W|_{i_1,i_m}} > 1 \tag{9.21}$$

Example 9.3.15. *The following system verifies the sufficient conditions for a pairwise unstable fully inhibiroty network, with* $\rho(A_1) = 1.2064$ *and* $\rho(A_2) = 0.0510$*:*

$$W = \begin{bmatrix} -2.0307 & -2.1786 & -8.1680 \\ -7.1715 & -1.4824 & -3.0371 \\ -1.6300 & -4.3597 & -0.7294 \end{bmatrix} \quad \boldsymbol{m} = \begin{bmatrix} 2.8224 \\ 1.5404 \\ 1.8834 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 1.3280 \\ 1.3738 \\ 0.7785 \end{bmatrix} \tag{9.22}$$

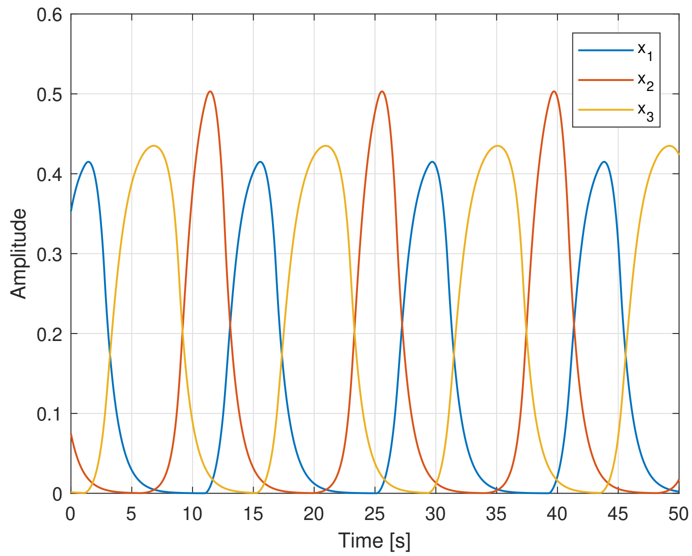


Figure 9.2: Temporal evolution of the network $(\boldsymbol{W}, \boldsymbol{m}, \boldsymbol{u})$ for nodes x_1, x_2 and x_3 verifying the sufficient conditions.

Corollary 9.3.16. *If a node does not belong to a valid cycle, then there exist one* $\boldsymbol{u}$ *for which it does not oscillate.*

Corollary 9.3.17. *If* $u \in C$ *then the condition is necessary and sufficient.*

Proof. If $\boldsymbol{u} \in C$ no node can have an equilibrium point in a region with some node in positive saturation, so the equilibrium points will be in one node in linear regions or in the all nodes in negative saturation region.

We remain to proof that if $\rho(A) < 1$ then $\nexists$ $\boldsymbol{u} \in C$ able to verify that $Au > u$, meaning that some inequalities will not be satisfied, and some equilibrium candidate for a region where only one node is in linear will have an stable equilibrium point.

Suppose $\boldsymbol{u} > 0$ exists.

Then, it is verified that:

$$\|A\boldsymbol{u}\| \leq \rho(A) \|\boldsymbol{u}\| \leq \|\boldsymbol{u}\| \tag{9.23}$$

Then as $\|A\boldsymbol{u}\| \leq \|\boldsymbol{u}\|$ it implies that at least there exists $u_i \geq A_{i,j} u_j$ meaning that the inequality will not be satisfied. $\square$

Example 9.3.18. *If* $\boldsymbol{u} \notin C$ *the conditions are only sufficient. The following system is a counter example for necessity if the input is not inside the bounded region. Note that the eigenvalues are* $\rho(A_1) = 0.1297$ *and* $\rho(A_2) = 0.7519$*, less than 1, but for the proposed input we have lack of stable equilibria:*

$$W = \begin{bmatrix} -0.8490 & -2.4585 & -4.4652 \\ -7.7335 & -2.1448 & -3.6943 \\ -2.9700 & -4.5271 & -2.4880 \end{bmatrix} \quad \boldsymbol{m} = \begin{bmatrix} 2.9578 \\ 1.9394 \\ 1.8413 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 7.9989 \\ 15.5028 \\ 11.1940 \end{bmatrix} \tag{9.24}$$

The temporal evolution of this system is:

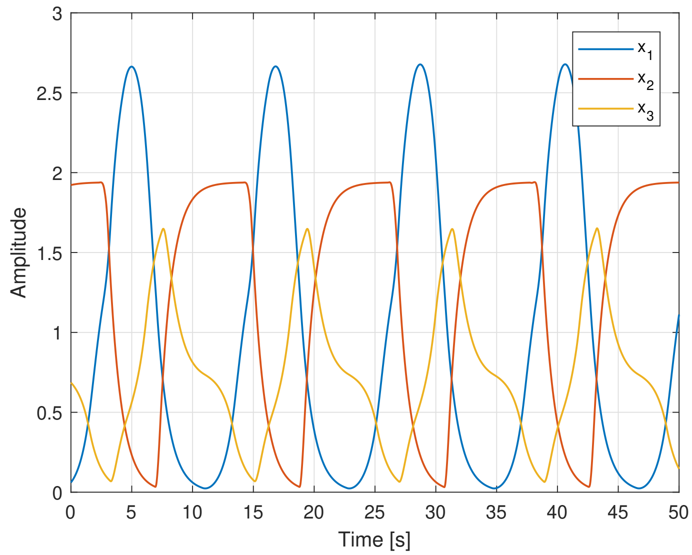


Figure 9.3: Temporal evolution of the network $(\boldsymbol{W}, \boldsymbol{m}, \boldsymbol{u})$ for nodes x_1, x_2 and x_3. Although the input u does not belong to a valid cycle the system oscillates.

Theorem 9.3.19. ***(Oscillations and valid cycle).***
If $\exists$ one node that belongs to all the valid cycles, assuming there exist at least one, then $\forall \boldsymbol{u} \in C$ the node will oscillate.

Proof. We will proof it by the counter-positive.
If there $\exists \boldsymbol{u} \in C$ for which the node does not oscillate, then the node does not belong to all the valid cycles.
If a node i does not oscillate, then it must be in positive or negative saturation. As $u_i < W_{i,i}m_i$ then it must be in negative saturation, because it cannot remain in positive saturation. If a node is in negative saturation $\forall t$ then it does not contribute at the oscillations of the other nodes, meaning that it is effectively as considering a m-1 network. For the m-1 network to oscillate it is necessary that there exists a valid cycle and this cycle will not include the i node. $\square$

Corollary 9.3.20. *If there $\exists!$ valid cycle, then the nodes in the cycle will be the ones that oscillates $\forall \boldsymbol{u} \in C$.*

Corollary 9.3.21. *Suppose $\boldsymbol{u}$ verifies the inequalities for a concrete cycle. The nodes of this cycle will oscillate.*

Chapter 10

Conclusion and future work

The main purpose of this thesis has been to study and bring understanding to the problem of the existence of oscillatory behavior in brain networks with bounded linear threshold dynamics. We have approach oscillatory behavior as lack of stable equilibria and numerically showed that this characterization can effectively bring explanation and origin to any oscillatory activity occurring in almost an eighty percent of the arbitrary excitatory-inhibitory networks.
Through an extensive geometrical analysis of the linear constrains intrinsic on the bounded dynamics, we have been able to derive three different sets of sufficient conditions on the network structure, that combined with restrictions on the inputs, can ensure lack of stable equilibiria. Furthermore, contributions on partial instability of certain regions have been made, arising concepts such as pairwise unstable networks or fully unstable networks. All of them, characterizations on the network structure that simplify the problem of lack of stable equilibria.
We have introduced and discussed the degenerate rate, numerically providing evidence of the competition that characterizes the oscillatory patterns. Both networks with arbitrary structure and with structures satisfying all the sets of sufficient conditions have been analyzed, however the conclusion remains the same for all of them: degenerate oscillations are a predominant phenomena. On the behalf of degeneration, we have been able to analytically provide characterization of it for fully inhibitory pairwise unstable networks, thanks to a proposed set of necessary and sufficient conditions on low inputs.
Although a significant characterization of networks with lack of stable equilibria has been made, specially through the so called geometrical approach, we still observe oscillatory behavior in networks with some stable equilibrium points, due to the existence of a limit cycle attractor. For future work, if a fully characterization of the oscillatory behavior is pursued, two main lines should be discussed: first, a focus on the structural conditions for the existence of limit cycles with stable equilibira in the network should be made, specially to give explanation to oscillatory behaviors involving only two nodes (due to the degenerate results found in the work) and, secondly, sets of sufficient conditions that omit our assumption on excitatory inestability (2).
In conclusion, the comprehension of the oscillatory in the brain is still far from being complete but with our results on the lack of stable equilibria we hope to have made a first step on this long way that needs to be traveled.

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Appendix A

Proof for the partial results

A.1 Conditions for existence of oscillations in a 1-excitatory - m-inhibitory network

Proof. The proof consists of two different parts, one focusing on the necessary conditions that $\boldsymbol{u}$ must verify in order for no region to contain its equilibrium candidate and the other one focusing on constructing a $\boldsymbol{u}$ for which, using the sufficient conditions, its equilibrium points will always fall outside the corresponding region if the excitatory node is not in linear state.

- **Necessary conditions**
  The necessary conditions arise with the regions where no node is in linear state. They are the same inequalities as expressed in (4.7), (4.9),(4.8) and (4.10) in a more compact form.

- **Existence of oscillations**
  To prove the existence of oscillations under the sufficient conditions a constructive proof will be presented by verifying that the $\boldsymbol{u}$ proposed in set A and set B are able to verify all the conditions. First note that, for the Set A, the sufficient conditions ($Sufficient\ condition$ 1) and ($Sufficient\ condition$ 2), using $(|W|_{1,1}-1)m_1 < |W|_{1,j}m_j$ are used to ensure room for u_1 and for u_j. The room for ($cond.\ 2A$) will come from:

$$\begin{aligned}
(\frac{(|W|_{1,1}-1)(|W|_{j,j}+1)}{|W|_{1,j}} - |W|_{j,1})m_1 <& -|W|_{j,1}m_1 + (1+|W|_{j,j})m_j) \\
\frac{(|W|_{1,1}-1)(|W|_{j,j}+1)}{|W|_{1,j}}m_1 <& (1+|W|_{j,j})m_j \\
(|W|_{1,1}-1)m_1 <& |W|_{1,j}m_j
\end{aligned}$$

  For the Set B, the sufficient conditions *Sufficient condition 1* and *Sufficient condition 2*, using $(|W|_{j,j}+1)m_j > |W|_{j,1}m_1$, are also used to ensure room for u_1 and for u_j.

  Now ,in heuristic way, for every $\boldsymbol{\sigma}$ it will be check the equilibrium does not fall into its corresponding region if the previous mentioned conditions hold.

- $\boldsymbol{\sigma} = (0, 0, \ldots, 0)$
  The union set of inequalities that must be verified for this $\boldsymbol{\sigma}$ are the ones described in (4.7) which will be immediately verified by $u_1 > 0$ because of (*cond.* $3A$) and (*cond.* $3B$) .

- $\boldsymbol{\sigma} = (s, 0, \ldots, 0)$
  The union set of inequalities for $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$ has the form of (4.8).
  For Set A, they will verified by the (*cond.* $2A$) as $u_j > -|W|_{j,1}m_1$ because of $\frac{(|W|_{1,1}-1)(|W|_{j,j}+1)}{|W|_{1,j}} > 0$.
  For Set B, also by (*cond.* $2B$) as $-|W|_{j,1}m_1 < 0 < u_j$

- $\boldsymbol{\sigma} = (0, \Sigma^{s,i}, \Sigma^{0,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$

  The inequalities that describe the valid region are the ones described in (4.9).
  For Set A the inequalities are trivially going to be verified thanks to (*cond.* $1A$) and (*cond.* $2A$), because of of $u_i < 0 \; \forall i \in \{2, \ldots, m+1\}$.
  For Set B, if $\exists \; \sigma_i = s$ fore some $i \in \{2, \ldots, m+1\}$ and $i \neq j$ then the inequalities will be verified as $u_i < 0$, thanks to (*cond.* $1B$). For the complementary case, the inequalities will be verified as $u_j < -|W|_{j,1}m_1 + (1 + |W|_{j,j})m_j < (1 + |W|_{j,j})m_j$ thanks to (*cond.* $2B$).

- $\boldsymbol{\sigma} = (s, \Sigma^{s,i}, \Sigma^{0,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$

  The inequalities that describe a valid region have the following form (4.10).
  By inspection of the inequalities, one can see that, there are two different cases: $\sigma_j = s$ and $\sigma_j = 0$. For both sets if $\sigma_j = 0$, then $u_i < -|W|_{i,1}m_1$, (*cond.* $1A$) and (*cond.* $1B$) will verify the inequalities.
  For the $\sigma_j = s$ then they are (*cond.* $2A$) and (*cond.* $2B$) that will make the inequalities to hold because $u_j < -|W|_{j,1}m_1 + (1 + |W|_{j,j})m_j < -|W|_{j,1}m_1 + (1 + |W|_{j,j})m_j + \sum_{\substack{i \neq j \\ \Sigma^{s,i}_{i,i}=1}} |W|_{j,i}m_i$.

- $\boldsymbol{\sigma} = (0, \Sigma^{0,i}, \Sigma^{l,i}) \quad s.t \mid \Sigma^{l,i} \mid + \mid \Sigma^{0,i} \mid = m$ and $\mid \Sigma^{l,i} \mid > 0$
  One of the following inequalities must be verified (4.11) for each $\Sigma^{l,i}$.
  For the Set A, the inequalities will be verified as a consequence of lemma (5.1.9).
  For the Set B, lemma (5.1.8), will guarantee that the inequalities are satisfied for every $\Sigma^{l,i}$ such that $|\Sigma^{l,i}| \geq 2$. For $|\Sigma^{l,i}| = 1$ there are two approaches:

  * $\sigma_j = l$
    The inequalities will be verified by (*cond.* $3B$) as $\frac{|W|_{1,j}}{(1+|W|_{j,j})} u_j < 0 < u_1$.

  * $\sigma_j \neq l$
    Following the proof of (5.1.9) the inequalities will be verified as $u_i < 0 \; \forall i \in \{2, \ldots, m+1$ and $i \neq j$.

- $\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$

The generalized inequalities that describes this region are (4.11).
For set A, as before,the inequalities will be verified as a consequence of lemma (5.1.9).
For the Set B, lemma (5.1.8), will guarantee that the inequalities are satisfied for every $\Sigma^{l,i}$ such that $|\Sigma^{l,i}| \geq 2$.
For $|\Sigma^{l,i}| = 1$ there are two approaches:

* $\sigma_j = l$
  The inequalities will be verified by $(cond.\ 2B)$ as $u_j < \min_{i\neq 1,j}(|W|_{j,i}m_i) < \sum_{\substack{i\neq j\\ \Sigma^{s,i}_{i,i}=1}} |W|_{j,i}m_i$.

* $\sigma_j \neq l$
  Following the proof of (5.1.9) the inequalities will be verified as $\forall i \in \{2, \ldots, m+1\}$ and $i \neq j$ then $u_i < 0 < \sum_{\substack{i\neq j\\ \Sigma^{s,i}_{i,i}=1}} |W|_{j,i}m_i$.

– $\boldsymbol{\sigma} = (s, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ and $\mid \Sigma^{s,i} \mid > 0$ and $\mid \Sigma^{l,i} \mid > 0$
The generalized inequalities that describes the valid $\boldsymbol{u}$ for these regions are (4.11).
For both Set A and Set B two possible cases applies:

* $j \in L$ *and* $|L| \geq 2$ *or* $j \notin L$ By respectively $(cond.\ 1A)$ and $(cond.\ 1B)$ and using the lemma (5.2.7) with a displaced origin the inequalities are verified.

* $j \in L$ *and* $|L| = 1$ Lets consider two possible sub-cases, inside this case:

· $\boldsymbol{\sigma} = (s, \boldsymbol{\Sigma}^0, l_j)$
For both Set A and Set B, it will be verified the inequality $(in.\ 1)$ from (4.11) which will have a form like:

$$u_1 < (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{|W|_{j,j}+1})m_1 + \frac{|W|_{1,j}}{|W|_{j,j}+1}u_j$$

Which will admit room for *Sufficient condition 1* and will hold thanks to $(cond.\ 3A)$ and $(cond.\ 3B)$.

· $\boldsymbol{\sigma} = (s, \boldsymbol{\Sigma}^0, \boldsymbol{\Sigma}^s, l_j)$
It will be verified by either inequality $(in.\ 1)$ or $(in.\ 3)$ which they will have a form like the following

$$u_1 < (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{|W|_{j,j}+1})m_1 + \frac{|W|_{1,j}}{|W|_{j,j}+1}u_j + \sum_{k\neq j,1\in S}\big(|W|_{1,k} - \frac{|W|_{1,j}|W|_{j,k}}{|W|_{j,j}+1})$$
$$u_j < -|W|_{j,1}m_1 + \sum_{k\neq j,1\in S}|W|_{j,k}m_k$$

By inspection one of them be always verified applying *Sufficient condition 3*.

□

A.2 Algorithm approach for oscillations in a 1-excitatory - m-inhibitory network

Proof. In order to proof that the algorithm will output an input $\boldsymbol{u}$ for which the system will have no stable equilibrium, each possible region $\boldsymbol{\sigma}$ is going to be checked proving that the equilibrium does not fall into it.

- $\boldsymbol{\sigma} = (0, 0, \ldots, 0)$
  The equilibrium will not fall into its region if $\exists\ u_i$ such that $u_i > 0$.
  For Set of conditions A, it will be verified by $u_1 > 0$.
  For Set B, it will be verified by either or $u_j > 0$ or $u_1 > 0$.

- $\boldsymbol{\sigma} = (s, 0, \ldots, 0)$
  The inequalities for $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$ for are of the following form of (4.8).
  For both sets A and B the inequalities will be verified by $u_j > -|W|_{k,1}m_1$, direct consequence of $(Cond.2A), (Cond.2B)$ lower bounds.

- $\boldsymbol{\sigma} = (0, \Sigma^{s,i}, \Sigma^{0,i}) \qquad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$

  The inequalities that describe the valid region are found at (4.9).
  It is easy to check that for every $\sigma_k = s$ the inequalities will look like:

$$u_k < (1 + |W|_{k,k})m_k + \sum_{\substack{t \neq 1,k \\ \Sigma^s_{t,t}=1}} W_{k,t}m_t \tag{A.1}$$

  But, in particular we have that, for every inhibitory node.

$$u_k < (1 + |W|_{k,k})m_k - W_{k,1}m_1 \tag{A.2}$$

  Because of the combination of the bounds $(Cond.\ 2A)$ and $(Cond.\ 1A)$ for the set A, and $(Cond.\ 2B)$ and $(Cond.\ 1B)$ for the set B.

- $\boldsymbol{\sigma} = (s, \Sigma^{s,i}, \Sigma^{0,i}) \qquad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$ and $\mid \Sigma^{s,i} \mid \geq 1$
  The inequalities that describe a valid region have the form of (4.10).
  Again, the combinations of the bounds $(Cond.\ 2A)$ and $(Cond.\ 1A)$ for the set A, and $(Cond.\ 2B)$ and $(Cond.\ 1B)$ for the set B will make the inequalities to be always verified. As $\mid \Sigma^{s,i} \mid \geq 1$, in $\boldsymbol{\sigma}$ that means there is always an inequality looking like

$$u_i < -|W|_{i,1}m_1 + (1+|W|_{i,i})m_i < -|W|_{i,1}m_1 + (1+|W|_{i,i})m_i + \sum_{\substack{t \neq 1,i \\ \Sigma^s_{t,t}=1}} W_{k,t}m_t \quad \forall i \in 2, \ldots, m+1\} \tag{A.3}$$

- $\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$
  The inequalities that describe a valid $\boldsymbol{u}$ for a $\boldsymbol{\sigma}$ of this specific form are (**??**).
  For the Set A , it is a direct implication of the lemma (5.1.9) and ($Cond.\ 4A$) for $O_s = 0$
  For the Set B, it is a direct implication of the lemma (5.1.10) and ($Cond.\ 4B$) for $O_s = 0$.

- $\boldsymbol{\sigma} = (s, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \quad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$ The generalized inequalities that describes the valid $\boldsymbol{u}$ for these regions are (4.11).
  We can see that the algorithm is creating a chain with a displaced origin O_1 (5.2.5) so we have to check that the conditions for j, the first node of the chain, and for the last one are held .
  For both Set A and Set B two possible cases applies:

  – $j \in L \ and \ |L| = 1$ Lets consider two possible sub-cases, inside this case:

    ∗ $\boldsymbol{\sigma} = (s, \boldsymbol{\Sigma}^0, l_j)$
      For both Set A and Set B, it will be verified the inequality ($in.\ 1$) from (4.11) which will have a form like:

$$u_1 < (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{|W|_{j,j} + 1})m_1 + \frac{|W|_{1,j}}{|W|_{j,j} + 1}u_j$$

      Which will admit room for *Sufficient condition 1* and will hold thanks to ($cond.\ 3A$) and ($cond.\ 3B$).

    ∗ $\boldsymbol{\sigma} = (s, \boldsymbol{\Sigma}^0, \boldsymbol{\Sigma}^s, l_j)$
      It will be verified by either inequality ($in.\ 0$) or ($in.\ 3$) which they will have a form like the following

$$u_1 < (1 - |W|_{1,1} + \frac{|W|_{1,j}|W|_{j,1}}{|W|_{j,j} + 1})m_1 + \frac{|W|_{1,j}}{|W|_{j,j} + 1}u_j + \sum_{k \neq j, 1 \in S} \big(|W|_{1,k} - \frac{|W|_{1,j}|W|_{j,k}}{|W|_{j,j} + 1})$$
$$u_j < -|W|_{j,1}m_1 + \sum_{k \neq j, 1 \in S} |W|_{j,k}m_k$$

      By inspection one of them be always verified applying *Sufficient condition 3*.

  – For $u_{\sigma_{m+1}}$ we have:

$$u_{\sigma_{m+1}} < \min_{k \neq 1, \sigma_{m+1}} (|W|_{\sigma_{m+1},k}m_k) < \sum_{k \neq 1, \sigma_{m+1}} |W|_{\sigma_{m+1},k}m_k \tag{A.4}$$

That way, the chain hypothesis will be verified and no region of these kind will contain its equilibrium candidate.

□

A.3 Sufficient conditions existence of non degenerate oscillations in 1-excitatory - m-inhibitory

Proof. Now let's consider a $\boldsymbol{u}$ in this region and check that $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$.
The proof will consists on considering each possible region $\boldsymbol{\sigma}$ and checking that the equilibrium does not fall into it if the $\boldsymbol{u}$ has that specific form, and giving the values of ε_i while doing so.

- $\boldsymbol{\sigma} = (0, 0, \dots, 0)$
  The equilibrium will not fall into its region if $\exists\ u_i$ such that $u_i > 0$, which will be immediately verified by $u_1 > 0$ (*cond.* 1).

- $\boldsymbol{\sigma} = (s, 0, \dots, 0)$
  The inequalities for $\boldsymbol{u} \in \mathbb{R}^N \backslash Y$ for are of the following form of (4.9) which will be verified by the (*cond.* 2).

- $\boldsymbol{\sigma} = (0, \Sigma^{s,i}, \Sigma^{0,i}) \qquad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$

  The inequalities that describe the valid region are the same ones as before, found at (4.9). They are trivially going to be verified by (*cond.* 2) and (*cond.* 3), because $u_i < 0\ \forall i \in \{2, \dots, m+1\}$.

- $\boldsymbol{\sigma} = (s, \Sigma^{s,i}, \Sigma^{0,i}) \qquad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid = m$
  The inequalities that describe a valid region have the form of (4.10).
  $\boldsymbol{u}$ will verify the inequalities given by nodes in positive saturation, thanks to the combination of *Sufficient condition 1* and conditions (*cond.* 2) and (*cond.* 3).

- $\boldsymbol{\sigma} = (0, \Sigma^{0,i}, \Sigma^{l,i}) \qquad s.t \mid \Sigma^{l,i} \mid + \mid \Sigma^{0,i} \mid = m$
  $\boldsymbol{\sigma} = (0, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \qquad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$
  This two cases work as the previous proof for the same type of $\boldsymbol{\sigma}$, being verified because $u_i < 0$ for $i \in \{2, \dots, m+1\}$.
- $\boldsymbol{\sigma} = (s, \Sigma^{l,i}, \Sigma^{0,i}, \Sigma^{s,i}) \qquad s.t \mid \Sigma^{s,i} \mid + \mid \Sigma^{0,i} \mid + \mid \Sigma^{l,i} \mid = m$

  Here we will distinguish different cases. First let $\mathbb{J}$ be the set of nodes in linear state, and let $\mathbb{L}^C \cap \mathbb{J}$ be the set of nodes in linear state that do not belong to the $\mathbb{L}$ set.

  - $\mathbb{L}^C \cap \mathbb{J} \neq \emptyset$
    As $j \in \mathbb{L}^C \cap \mathbb{J}$ and $u_j < -|W|_{i,1} m_1$ some of the inequalities of the form (*in.* 1) or (*in.* 2) will be held.
  - $\mathbb{L}^C \cap \mathbb{J} = \emptyset$
    In here we have to distinguish with another two possible cases.

- $|l| = 1$
  If only the excitatory node is positive saturation state then the inequality that will be verified will be:
  $$(0 > u_i) > -W_{i,1}m_1 + (W_{i,i}+1)m_i \tag{A.5}$$
  Thanks to *Sufficient condition 2*.
  If there is another node in positive saturation (k), then by *Sufficient condition 3* we have that:
  $$(u_i < 0) < -W_{i,1}m_1 + W_{i,k}m_k \tag{A.6}$$

- $|l| \geq 2$ We will focus on the first inequality, involving the excitatory node.
  The form of this first inequality is :
  $$u_1 < \sum_{j\in 1,\dots,m+1} \big(- sg(W_{1,j})|W|_{1,j}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^l)_{kk}u_k + \underbrace{K^l_{j,k}(\Sigma^m)_{kk}m_k)\big) + m_1}_{(1)}$$
  Considering only the (1) term, we have:
  $$\begin{aligned}
  (1) = &\sum_{j\in 1,\dots,m+1} \big(- sg(W_{1,j})|W|_{1,j}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^m)_{kk}m_k)\big) + m_1 \\
  &= -|W|_{1,1}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^m)_{kk}m_k) \\
  &\quad + \sum_{j\in 2,\dots,m+1} \big(+ |W|_{1,j}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^m)_{kk}m_k)\big) + m_1 \\
  &= -|W|_{1,1}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^m)_{kk}m_k) + \sum_{j\in 2,\dots,m+1} (A_j) + m_1 \\
  &> -|W|_{1,1}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^m)_{kk}m_k) - \sum_{j\in 2,\dots,m+1} \mathbb{I}_{\{sg(A_j)=+1\}}A_j + m_1 \\
  &> \min_{\substack{\Sigma^l,\Sigma^m \\ (\Sigma^m)_{1,1}=1}} \Big(- |W|_{1,1}(\sum_{k\in 1,\dots,m+1} (K^l_{j,k}(\Sigma^m)_{kk}m_k) - \sum_{j\in 2,\dots,m+1} \mathbb{I}_{\{sg(A_j)=+1\}}A_j \Big) + m_1 \\
  &\underset{\textit{Sufficient condition 4}}{>} 0
  \end{aligned}$$
  Thus implies that there will always exists sufficiently small u_i $i \in \{2,\dots,m+1\} \cap \mathbb{J}$ for the inequalities to hold.

  Then, although other forms can be found, some possible boundaries for ε_i are going to be proposed.
  So let $B^l = \big((diag(A^l) + [A^l - diag(A^l)]^+(\boldsymbol{I} - \Sigma^l)\boldsymbol{m}\big)_1$. For *Sufficient condition 4* we have that $m_1 - B^l > 0$.
  Taking into account the inequality, and letting s be the set of positive saturated nodes with

the exitatory node in positive saturation, one has that:

$$\begin{aligned} u_1 + A^l(\Sigma^l \boldsymbol{u} + \Sigma^s \boldsymbol{m}) &< m_1 \\ u_1 &< m_1 - A^l(\Sigma^l \boldsymbol{u} + \Sigma^s \boldsymbol{m}) \\ m_1 - A^l(\Sigma^l \boldsymbol{u} + \Sigma^s \boldsymbol{m}) &> m_1 - B^l - ([A^l]^-)\Sigma^l \boldsymbol{u} \end{aligned} \tag{A.7}$$

So for giving a sufficient small boundary for each u_i with $i \in \mathbb{L}$ and for u_1, for which $\boldsymbol{u}$ will verify the previous inequalities and the other ones mentioned in the proof it is needed tht:

$$\begin{cases} \varepsilon_1 = \min\Big(\big(\min\limits_{i\in\mathbb{L}}(1 - |W|_{1,1})m_1 + |W|_{1,i}m_i\big), \big(\min\limits_{\substack{\forall l \subseteq \mathbb{L} \\ |L|\geq 2}} \dfrac{m_1 - B^l}{\#[(A^l\Sigma^l)^-_{1,:}] + 1}\big)\Big) \\ \varepsilon_i = \min\Big(\big(\min\limits_{i\in\mathbb{L}} -W_{i,1}m_1 + (W_{i,i}+1)m_i\big), \big(\min\limits_{\substack{\forall l \subseteq \mathbb{L} \\ |L|\geq 2}} \dfrac{m_1 - B^l}{(\#[(A^l\Sigma^l)^-_{1,:}] + 1)A^l_{1,i}}\big)\Big) \qquad \forall i \in \mathbb{L} \end{cases} \tag{A.8}$$

□

Appendix B

Codes

The following section contains only main codes used during the development of the work. This includes the simulations for the determination of the validity lack of equilibria approach, the algorithm to check is a temporal evolution oscillates and the determination if there is stable equilibria, and the algorithm approach to the sets of sufficient conditions.

B.1 Validity of the lack of equilibria approach

```
%% PROXY SIMULATION

% Parameters for random generation of the system matrix
m_max = 10;
a_max = 7;
d_max = 7;
e_max = 4;
i_max = 4;

% Objective number of oscillations aimed to find
Oscilations_min = 50;

% Study parameters
% Oscillation from lack of equilibria
NS = zeros(e_max,i_max);
% Oscillation from limit cycles
S =  zeros(e_max,i_max);
% Region of attraction
Percentage_X0 = zeros(e_max,i_max);
% Number of cases considered
Total = zeros(e_max,i_max);
```

```
% Degerenate rate
Degenerate_Rate_arbitrary = zeros(e_max,i_max);

%Parameter to find the region of attraction
it_x0_local_max = 100;

 for e = 1:e_max
 for i = 1:i_max
 n = e;
 m = i;
 N = n+m;
 tau = ones(N,1);
 while(NS(n,m) + S(n,m) < Oscilations_min)
     Total(n,m) = Total(n,m) + 1;

     % Random system generation
     W = W_generate_random(a_max,d_max, n,m) ;
     m_vec = Random_Vector(n+m,0.01,m_max);
     mean_W = mean(mean(abs(W)));
     mean_m= mean(m_vec(:));
     u = Random_Vector(n+m,-mean_W*mean_m*n,mean_W*mean_m*m);

     % Random inital point
     x0 = Random_Vector(n+m,0.01,m_max);

     % Check if the system oscillates
     Nodes = Check_Oscillations(W,u,tau,m_vec,N, x0);

     % If oscialltes then:
     if(length(Nodes) >=  2)
         %Compute the degenerate rate
         Degenerate_Rate_arbitrary(n,m) =
   Degenerate_Rate_arbitrary(n,m) + length(Nodes);

         % Check if there exist stable equilibria
         Bool_Stable = Check_Limit_Cycle(W,m_vec,u,N);
         if(~Bool_Stable)
             NS(n,m) = NS(n,m) + 1;
         else
             S(n,m) = S(n,m) +1;

             % Computation of the region of attraction
```

```
            for it_x0_local = 1:it_x0_local_max
            x0 = Random_Vector(n+m,0.01,m_max);
            Nodes = Check_Oscillations(W,u,tau,m_vec,N, x0);
            if(length(Nodes) >= 2)
               Percentage_X0(n,m) = Percentage_X0(n,m) +1;
            end
            end
        end
    end
end
end
end
```

B.2 Existence of oscillatory behavior check

```
function [Nodes] = Check_Oscillations(W,u,tau,m_vec,N,x0 )

% Initialize an empty vector for the nodes that are
% oscillating
Nodes = [];

% For the chosen intial condition let it evolve thorugh time
tstep = 0.1;            % Sampling period
T = 40;                 % Length of signal
tspan = 0:tstep:T;      % Time vector
options = odeset('RelTol', 1e-3, 'AbsTol', 1e-5);
[~, x] = ode45(@(t, x)LT_sim_odefun(t, x, W, u, tau, m_vec), tspan, x0, options);
x0 = x(end,:);

% From the ending point of the last evolution, do a more precise
% simulation and pick the mean value of it in order to try to avoid
% false positives

T = 10;
tstep = 0.1;
tspan = 0:tstep:T;
Fs = 1/tstep;
L = T/tstep;
```

```
options = odeset('RelTol', 1e-4, 'AbsTol', 1e-6);
[~, x] = ode45(@(t, x)LT_sim_odefun(t, x, W, u, tau, m_vec), tspan, x0,options );
x0 = mean(x(:,:));

% Do the final simulation of time evolution to check if it oscillates

T = 5;
tstep = 0.05;
tspan = 0:tstep:T;
Fs = 1/tstep;
L = T/tstep;
options = odeset('RelTol', 1e-5, 'AbsTol', 1e-7);
[~, x] = ode45(@(t, x)LT_sim_odefun(t, x, W, u, tau, m_vec), tspan, x0,options );

% Do a first threshold of oscillations to avoid a stable equilibira
mean_X = mean(x(:, :));
OSCILLATION_CANDIDATES = zeros(N,1);
Max_X = max(x(:,:));
Min_X = min(x(:,:));
for i = 1:N
    if ( abs(mean_X(i)) > 0.01 && abs(mean_X(i)) < m_vec(i) - 0.01 ...
            && abs(Max_X(i)- Min_X(i)) >0.1)
        OSCILLATION_CANDIDATES(i) = 1;
    end
end

% Compute the Xreg parameter for each variable
epsilon = 0.1;
xf = fft(x(:, :) - mean(x(:, :)));
P2 = abs(xf);
Points = T/tstep;
P1 = P2(1:(Points)/2+1,:);
P1(2:end-1) = 2*P1(2:end-1);
f = Fs*(0:(Points/2))/Points;
[Xf_max, I] = max(abs(P1(1:end, : )));
f_obj = f(I);
f_obj_min = f_obj*(1-epsilon);
```

```
f_obj_max = f_obj*(1+epsilon);
I_min = -1*ones(1,N);
I_max = -1*ones(1,N);
for t = 1:length(f_obj)
    for r = 1:length(P1(:,t))
        if (I_min(t) == -1 && f_obj_min(t)>= f(r))
            I_min(t) = r;
        end
        if (I_max(t) == -1 && f_obj_max(t)<= f(r))
            I_max(t) = r;
        end
    end
end

% Check the nodes oscillating
Chi = ones(1,N);
for t = 1:N
    if (Xf_max(t) < 2 || OSCILLATION_CANDIDATES(t) == 0  )
        Chi(t) = 0;
    else
        Chi(t) = Xf_max(t)/ max(abs(P1(I_min(t), t)),abs(P1(
   I_max(t), t)));
    end
end

for it_l = 1:N
    if (~isnan(Chi(it_l)) && Chi(it_l)>2 &&
   OSCILLATION_CANDIDATES(it_l) == 1 )
        Nodes = [Nodes,it_l];
    end
end

end
```

B.3 Origin of the oscillatory behavior check

```
function Bool_Stable = Check_Limit_Cycle(W,m_vec,u,N)

% Initialize supposing lack of stable equilibria
Bool_Stable = false;
Y= true;

% For every switching region check
for i = 0:3^N-1
    sigma = str2num(dec2base(i, 3, N)');
    x_star = (eye(N) - diag(sigma == 1) * W) \ ...
        (diag(sigma == 1) * u + diag(sigma == 2) * m_vec);
    A_mat = W * x_star + u;
    Y= true;

    % Check if the equilibrium candidate is inside
    for T = 1:N
        if (sigma(T) == 2 && Y)
            Y = (A_mat(T)>= m_vec(T));
        end
        if (sigma(T) == 0 && Y)
             Y = (A_mat(T)<= 0);
        end
        if (sigma(T) == 1 && Y)
            Y = ((A_mat(T)>= 0)&&(A_mat(T)<= m_vec(T)));
        end
    end

    % In case of being inside, check if it is stable
    if (Y == true)
        if all(real(eig((-eye(N) + diag(sigma == 1) * W))) < 0)

            % If both conditions are staisfied, return true
            Bool_Stable = true;
        end
    end
end
```

B.4 Algorithm for sufficient conditions

```
function u = Algorithm(W,m_vec,n,m,N,j, Epsilon, u_0)

u = u_0;
W_abs = abs(W(:,:));
P= randperm(m);
P = P +n;
aux_i = (find(P(:) == j));

% Picking an arbitrary chain with the first node at the begining
P([1, aux_i] ) = P([aux_i, 1] );

I_W = diag(ones(N,1)) - W;
for it_j = 1:length(P)
j1 = P(it_j);
for i = 0:2^(N) -1
    SS= str2num(dec2base(i, 2, N)');
    for it_w = (it_j+1):length(P)
        w =P(it_w);
        if(SS(j1) == 0 && SS(w) == 0)
        O_s = I_W*(diag(SS))*m_vec;
        v = u - O_s;
        PIjw= [v(j1), v(w)];
        if (PIjw(1) > 0 && PIjw(2) > 0 )
            MM = min ((W_abs(w,w) +1 )/W_abs(j,w),...
                (W_abs(w,j))/(W_abs(j,j) +1 ) );

            % Compute the minim value
            for it_t = it_j:length(P)
            t = P(it_t);
            MM = min (MM, I_W(w,t)/I_W(j,t));
            end

            % Take out of the cone throught the -w direction
            if (PIjw(2)/PIjw(1) > MM)
            PIjw(2) = PIjw(1)* MM- Epsilon;
            end
        elseif(PIjw(1) < 0 && PIjw(2) > 0)
            % Take out of the cone throught the -w direction
            PIjw(2) = -Epsilon;
```

```
            end
            u(w) = PIjw(2) + O_s(w) ;
            end
      end
end
end

end
```